Notes on Complex Linear Phase FIR Filters

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1 Introduction

Most analyses of linear phase FIR filters assume real coefficients. Here we deal with the more general case of complex coefficients. The frequency response of a (generalized) linear phase system can be expressed as

$$H(\omega) = e^{j\alpha\omega} e^{j\beta} B(\omega), \tag{1}$$

where $B(\omega)$ is a real function of frequency. There is a phase contribution from each of the three factors in this equation: the first gives is a linear phase, the second gives a constant phase, and the third contributes a phase of either 0 or π. In this report, we will derive the symmetry conditions on the coefficients of an FIR filter such that the frequency response is of this form. Furthermore we derive a factorization of the system response for the different forms of symmetry.

2 Symmetry Conditions

2.1 Simplified Case

First consider a simplified case. The filter $G(z)$ will be causal with a non-zero first coefficient $g[0]$ and a non-zero last coefficient $g[N-1]$. Consider the frequency response,

$$G(\omega) = e^{j\alpha\omega} B(\omega), \tag{2}$$

where $B(\omega)$ is purely real. We may prefer to think of the backwards relation to Eq. (2),

$$B(\omega) = e^{-j\alpha\omega} G(\omega). \tag{3}$$

In this form, we are trying to find the requirements on the coefficients of $G(\omega)$ such that $B(\omega)$ becomes purely real.

To accommodate both even and odd numbers of coefficients, we will consider the up-sampled response[1]

$$B(z^2) = z^{-2\alpha} G(z^2). \tag{4}$$

[1] This artifice is being introduced to solve the problem that for an even number of coefficients, we cannot shift the FIR time response corresponding to $G(z)$ so that it becomes symmetrical about zero.

The up-sampling operation inserts zero values between the original samples of $G(z)$, giving an odd-length sequence of length $2N-1$. The samples of the response corresponding to $G_2(z^2)$ are

$$g_2[n] = \begin{cases} g[n/2], & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{5}$$

Consider a sequence $x[n]$ with frequency response $X(\omega)$. It is straightforward to show that

$$\tfrac{1}{2}(x[n] + x^*[-n]) \leftrightarrow \mathrm{Re}[X(\omega)] \tag{6}$$

This implies, if $X(\omega)$ is to be purely real, then $x[n] = x^*[-n]$. Identifying $X(z)$ with $B(z^2)$, then for the frequency response $B(2\omega)$ to be purely real, the time response corresponding to $B(z^2)$ has to have conjugate symmetry,

$$b_2[n] = b_2^*[-n]. \tag{7}$$

If 2α in Eq. (4) is an integer, since $G(z^2)$ is finite length (length $2N-1$), then the time responses of $B(z^2)$ and $G(z^2)$ are just shifted versions of each other. The term $z^{-2\alpha}$ in Eq. (4) represents the time shift. Since $g[0]$ (and hence $g_2[0]$) is by assumption non-zero, the only permissible shift is that which shifts $g_2[0]$ to become $b_2[-N-1]$. That shift (in samples) is[2]

$$2\alpha = -(N-1). \tag{8}$$

The time response $b_2[n]$ will have $N-1$ samples to the left of zero and $N-1$ samples to the; $g[n]$ is found from $g_2[n]$ by selecting every second sample, or equivalently by selecting every second sample of $b_2[n]$,

$$g[n] = b_2[2n-(N-1)]. \tag{9}$$

Using Eq. (7),

$$\begin{aligned} g[(N-1)-n] &= b_2[(N-1)-2n] \\ &= b_2^*[2n-(N-1)] \\ &= g^*[n]. \end{aligned} \tag{10}$$

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[2] We need to eliminate the possibility that the shift is non-integer. Since $G(z^2)$ is FIR, except for a scale factor, it is completely specified by its zeros. The shift $z^{2\alpha}$ can only introduce singularities (poles or zeros) at $z=0$. The response $B(z^2)$ will have the same zeros (other than those affected by the shift). Thus it is also FIR and can only be a shifted version of $G(z^2)$. Ergo 2α must be integer.

This is the symmetry requirement on $g[n]$. The corresponding frequency response is linear phase with $\alpha = -(N-1)/2$,

$$G(\omega) = e^{-j\omega(N-1)/2} B(\omega), \tag{11}$$

where $B(\omega)$ is real.

2.2 Constant Phase Term

We can add a constant phase term to Eq. (4)

$$B(z^2) = e^{-j\beta} z^{-2\alpha} G(z^2). \tag{12}$$

The equivalent form in terms of time responses is

$$b_2[2n-(N-1)] = e^{-j\beta} g[n]. \tag{13}$$

Then using the symmetry of $b_2[n]$ (Eq. (7)),

$$g[N-1-n] = e^{-j2\beta} g^*[n]. \tag{14}$$

From this result, the magnitudes of the coefficients must satisfy

$$\left|g[N-1-n]\right| = \left|g[n]\right|. \tag{15}$$

The z-transform version of Eq. (14) is

$$G(z) = z^{-(N-1)} e^{-j2\beta} G^*(1/z^*). \tag{16}$$

The values of the phase β are constrained by the number and type of coefficients in the filter as shown below.

2.2.1 Phase term for complex coefficients

1. If $\beta = 0$ or $\beta = \pi$, the coefficients are complex conjugates of each other,

$$g[N-1-n] = g^*[n]. \tag{17}$$

   If the number of coefficients is odd, there is a middle coefficient which must be its own conjugate, i.e. the middle coefficient must be real.

2. If $\beta = \pm\pi/2$, the coefficients are negative conjugates of each other,

$$g[N-1-n] = -g^*[n]. \tag{18}$$

   If the number of coefficients is odd, there is a middle coefficient which must imaginary.

For other values of β, if the number of coefficients is odd, the middle coefficient must be of the form $re^{-j\beta}$, where r is real. However, note that in some cases, as described below, r is constrained to be zero.

These relationships are illustrated in Table 1.

Table 1 Coefficient symmetries for complex coefficients (*a* and *b* are complex; *r* is real)

β	N even	N odd
0 or π	$\{a \quad b \quad b^* \quad a^*\}$	$\{a \quad b \quad r \quad b^* \quad a^*\}$
$\pm\pi/2$	$\{a \quad b \quad -b^* \quad -a^*\}$	$\{a \quad b \quad jr \quad -b^* \quad -a^*\}$
other	$e^{-j\beta}\{ae^{j\beta} \quad be^{j\beta} \quad (be^{j\beta})^* \quad (ae^{j\beta})^*\}$	$e^{-j\beta}\{ae^{j\beta} \quad be^{j\beta} \quad r \quad (be^{j\beta})^* \quad (ae^{j\beta})^*\}$

2.2.2 *Phase term for real coefficients*

For the coefficients to be real, the term $e^{-j2\beta}$ in Eq. (14) must be real, with the implication that only $\beta = 0$, $\beta = \pi$, and $\beta = \pm\pi/2$ are possible values for the constant phase term. If the number of coefficients is odd and $\beta = \pm\pi/2$, the middle coefficient must be zero. These relationships are illustrated in Table 2.

Table 2 Coefficient symmetries for real coefficients

β	N even	N odd
0 or π	$\{a \quad b \quad b \quad a\}$	$\{a \quad b \quad c \quad b \quad a\}$
$\pm\pi/2$	$\{a \quad b \quad -b \quad -a\}$	$\{a \quad b \quad 0 \quad -b \quad -a\}$

2.3 Shifted Filter Coefficients

Finally we will shift the filter response by M samples,

$$H(z) = z^{-M} G(z). \tag{19}$$

The first non-zero coefficient is now at $n = M$ and the last non-zero coefficient is at $n = N + M - 1$. The symmetry conditions on the coefficients of the shifted filter are (from Eq. (14))

$$h[2M + N - 1 - n] = e^{-j2\beta} h^*[n]. \tag{20}$$

In terms of the transfer function,

$$H(z) = z^{-2M-N+1} e^{-j2\beta} H^*(1/z^*). \tag{21}$$

We can summarize the results for this most general case: the linear phase factor is constrained to be of the form $\alpha = M + (N-1)/2$, where M (position of the first non-zero coefficient) and N (extent of the coefficients) are integer values.

3 Singularities of the Filter Response

The filter $G(z)$ is causal with first coefficient at $n = 0$. It has $N-1$ poles at the origin and $N-1$ zeros in the finite z-plane ($0 < z < \infty$) The filter $G(z)$ is shifted by M samples to form $H(z)$. The singularities of $H(z)$ are enumerated below.

$M \ge 0$

There are now $M + N - 1$ poles at the origin, the same $N-1$ zeros in the finite z-plane ($0 < z < \infty$), and M zeros at $z = \infty$.

$M < 0$, $N - 1 - |M| \ge 0$

There are now $N - 1 - |M|$ poles at the origin, the same $N-1$ zeros in the finite z-plane ($0 < z < \infty$), and $|M|$ poles at $z = \infty$.

$M < 0$, $N - 1 - |M| < 0$

For this case, there are $|M| - (N-1)$ zeros at the origin, the same $N-1$ zeros in the finite z-plane ($0 < z < \infty$), and $|M|$ poles at $z = \infty$.

3.1 Zero Symmetries: Complex Coefficients

From Eq. (21), if $z_o = re^{j\phi}$ is a zero of $H(z)$, then $1/z_o^* = (1/r)e^{j\phi}$ is also a zero. Zeros on the unit circle can appear singly. The zero symmetries for the finite z-plane ($0 < z < \infty$) zeros are illustrated in Fig. 1.

3.2 Zero Symmetries: Real Coefficients

For real coefficients, the finite z-plane ($0 < z < \infty$) zeros must also appear in complex conjugate pairs. Thus if $z_o = re^{j\phi}$ is a zero of $H(z)$, then $z_o^* = re^{-j\phi}$, $1/z_o^* = (1/r)e^{j\phi}$, and $1/z_o = (1/r)e^{-j\phi}$ are also zeros. Complex zeros off the unit circle appear in fours. Complex zeros

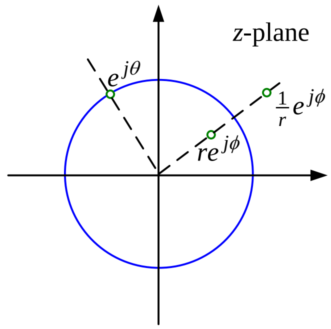

Fig. 1 Zero symmetries for complex coefficients

on the unit circle appear in pairs. Real zeros off the unit circle appear as pairs. Real zeros at ± 1 can appear singly. The zero symmetries are illustrated in Fig. 2.

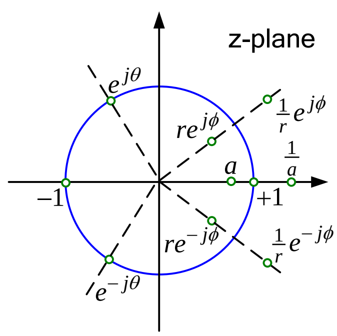

Fig. 2 Zero symmetries for real coefficients

3.3 Linear Phase Term

For $\beta = \pm\pi/2$, the transfer function will have an odd number of zeros at $z = 1$. These zeros give a phase jump of π degrees at zero frequency. This means that the phase response will jump from $\pi/2$ to $-\pi/2$ or vice versa, resulting in an overall phase response that is an odd function of frequency.

3.4 Constrained Zeros

For certain configurations, the filter response has zeros that are constrained to be at $z = 1$ and/or $z = -1$. Consider the symmetry requirements for a symmetric filter with an N-term impulse response (Eq. (16) repeated here)

$$G(z) = z^{-(N-1)} e^{-j2\beta} G^*(1/z^*). \tag{22}$$

We can then write

$$G(z) = \tfrac{1}{2} e^{-j\beta} \left[e^{j\beta} G(z) + z^{-(N-1)} e^{-j\beta} G^*(1/z^*) \right]. \tag{23}$$

We will examine this equation for $z = \pm 1$.

3.4.1 Zeros at $z = 1$

For $z = 1$,

$$\begin{aligned} G(1) &= \tfrac{1}{2} e^{-j\beta} \left[e^{j\beta} G(1) + e^{-j\beta} G^*(1) \right] \\ &= e^{-j\beta} \operatorname{Re}[e^{j\beta} G(1)]. \end{aligned} \tag{24}$$

The response at $z = 1$ will be zero ($G(z)$ has a zero at $z = 1$) if the coefficients are real and β is an odd multiple of $\pi/2$.

3.4.2 Zeros at $z = -1$

For $z = -1$,

$$\begin{aligned} G(-1) &= \tfrac{1}{2} e^{-j\beta} \left[e^{j\beta} G(1) + (-1)^{-(N-1)} e^{-j\beta} G^*(-1) \right] \\ &= \begin{cases} e^{-j\beta} \operatorname{Re}[e^{j\beta} G(-1)], & N \text{ odd}, \\ e^{-j\beta}\, j \operatorname{Im}[e^{j\beta} G(-1)], & N \text{ even}. \end{cases} \end{aligned} \tag{25}$$

For N odd, $G(-1)$ will be zero ($G(z)$ has a zero at $z = -1$) if the coefficients are real and β is an odd multiple of $\pi/2$. For N odd, $G(-1)$ will be zero if the coefficients are real and β is a multiple of π.

3.4.3 Real Coefficients

Specializing to real coefficients, we can write the frequency response as

$$G(\omega) = e^{-j\omega(N-1)/2} \tilde{Q}(\omega) B^0(\omega), \tag{26}$$

where $B^0(\omega)$ is a real-valued response, the so-called zero-phase factor, and $\tilde{Q}(\omega)$ is a fixed response. The factor $\tilde{Q}(\omega)$ is either purely real or purely imaginary. In the latter case, we can define a phase factor $e^{j\pi/2}$ to be lumped with the linear-phase term (giving a so-called generalized linear phase). Then the overall zero-phase response can be written as

$$H^0(\omega) = Q^0(\omega) B^0(\omega). \tag{27}$$

Appendix A Symmetric FIR Filters

Consider an FIR filter with N real coefficients,

$$G(z) = \sum_{n=0}^{N-1} g[n] z^{-n}. \tag{28}$$

An even-symmetric or odd-symmetric filter satisfies the following equation,

$$g[n] = \pm g[N-1-n], \tag{29}$$

where the + sign applies to even-symmetry and the – sign applies to odd-symmetry. In z-transform notation,

$$H(z) = \pm z^{-(N-1)} H(z^{-1}). \tag{30}$$

A.1 Root Factors

There are four filter types to consider: even- and odd-symmetric, and even and odd numbers of coefficients. Write $H(z)$ as follows,

$$H(z) = \frac{1}{2}[H(z) \pm z^{-(N-1)} H(z^{-1})]. \tag{31}$$

Consider $H(z)$ evaluated at $z = 1$,

$$H(1) = \frac{1}{2}[1 \pm 1^{-(N-1)}]H(1). \tag{32}$$

For the – sign (odd-symmetric), $H(1) = 0$, (any N). Now consider $H(z)$ evaluated at $z = -1$,

$$H(-1) = \frac{1}{2}[1 \pm (-1)^{-(N-1)}]H(-1). \tag{33}$$

For the + sign (even-symmetric), $H(-1) = 0$, if N is even. For the – sign (odd-symmetric), $H(-1) = 0$, if N is odd.

Noting the fixed roots found above, $H(z)$ can be factored as follows,

$$H(z) = Q(z)B(z), \tag{34}$$

where $Q(z)$ has only roots at $z = 1$ and/or $z = -1$. The term $B(z)$ is reduced in order, and as will be shown shortly, is even-symmetric with an odd number of coefficients. The results are summarized in the table below.

Type	$H(z)$	$Q(z)$	No. Coef. $B(z)$
I	N odd even-symmetric	1	N
II	N even even-symmetric	$1+z^{-1}$	$N-1$
III	N odd odd-symmetric	$1-z^{-2}$	$N-2$
IV	N even odd-symmetric	$1-z^{-1}$	$N-1$

The number of coefficients in $Q(z)$ is $N-M$, where M is the number of coefficients in $B(z)$ and the symmetry in $Q(z)$ can be expressed as

$$Q(z) = \pm z^{-(N-M)} Q(z^{-1}). \tag{35}$$

The upper sign applies for $H(z)$ even-symmetric, while the lower sign applies for $H(z)$ odd-symmetric. Then

$$\begin{aligned} B(z) &= H(z)/Q(z) \\ &= \pm z^{-(N-1)} H(z^{-1})/Q(z) \\ &= \pm z^{-(N-1)} Q(z^{-1}) B(z^{-1})/Q(z) \\ &= \pm z^{-(N-1)} [\pm z^{N-M} Q(z)] B(z^{-1})/Q(z) \\ &= z^{-(M-1)} B(z^{-1}). \end{aligned} \tag{36}$$

In this equation, there are just two cases: $H(z)$ even-symmetric (use the plus signs) or $H(z)$ odd-symmetric (use the minus signs). This shows that $B(z)$ is always even-symmetric. It always has an odd number of coefficients as seen from **Error! Reference source not found.**.

A.2 Frequency Response

The frequency response of a linear-phase filter can be expressed as

$$\begin{aligned} H(\omega) &= Q(\omega)B(\omega) \\ &= e^{-j\omega(N-M)/2} \tilde{Q}(\omega) e^{-j\omega(M-1)/2} B^0(\omega) \\ &= e^{-j\omega(N-1)/2} \tilde{Q}(\omega) B^0(\omega), \end{aligned} \tag{37}$$

where the response $B^0(\omega)$ can be expressed as

The fixed response $\tilde{Q}(\omega)$ is given in

Type	$H(z)$	$\tilde{Q}(\omega)$
I	N odd even-symmetric	1
II	N even even-symmetric	$2\cos(\omega/2)$
III	N odd odd-symmetric	$2j\sin(\omega)$
IV	N even odd-symmetric	$2j\sin(\omega/2)$

Table 3 The response $\tilde{Q}(\omega)$

$$
\begin{aligned}
B^0(\omega) &= \sum_{n=0}^{M-1} b[n] e^{-j\omega(n-(M-1)/2)} \\
&= b[(M-1)/2] + 2 \sum_{n=(M-1)/2+1}^{M-1} b[n]\cos(\omega(n-(M-1)/2)) \\
&= b[(M-1)/2] + 2 \sum_{n=1}^{(M-1)/2} b[n+(M-1)/2]\cos(\omega n) \\
&= \sum_{n=0}^{(M-1)/2} \tilde{b}[n]\cos(\omega n),
\end{aligned}
\tag{38}
$$

with the obvious definition of $\tilde{b}[n]$. This response is the *zero-phase response*. It is to be noted that $B^0(\omega)$ is a real function and can take on positive or negative values. The zero-phase response can be expressed as a Chebyshev polynomial in $x = \cos(\omega)$,

$$
B^0(\omega) = \sum_{n=0}^{(M-1)/2} \alpha_n (\cos(\omega))^n. \tag{39}
$$

The filter design program works with this transformed polynomial.

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Type	$H(z)$	$\tilde{Q}(\omega)$
I	N odd even-symmetric	1
II	N even even-symmetric	$2\cos(\omega/2)$
III	N odd odd-symmetric	$2j\sin(\omega)$
IV	N even odd-symmetric	$2j\sin(\omega/2)$

As given, this is not a zero-phase response.

The j factor associated with the response can be taken out and lumped with the linear-phase term to give a generalized linear-phase term,

$$P(\omega) = e^{j\phi(\omega)}, \quad (40)$$

where the phase is

$$\phi(\omega) = \begin{cases} -\omega\dfrac{N-1}{2}, & \text{even-symmetric filter,} \\ -\omega\dfrac{N-1}{2} + \dfrac{\pi}{2}, & \text{odd-symmetric filter.} \end{cases} \quad (41)$$