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Real and Complex Linear-Phase FIR Filters

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1 Introduction

Most analyses of linear-phase FIR filters assume real coefficients. Here we start with an analysis of the more general case of complex coefficients and later specialize those results to real coefficients. The frequency response of a (generalized) linear-phase system can be expressed as

$$H(\omega) = e^{-j\alpha\omega} \, e^{j\beta} \, B(\omega), \tag{1}$$

where $B(\omega)$ is a real function of frequency. There is a phase contribution from each of the three factors in this equation: the first gives is a linear-phase (corresponding to a delay of α), the second gives a constant phase, and the third is a real response which contributes a frequency dependent phase of either 0 or π. In this report, we will derive the symmetry conditions on the coefficients of an FIR filter such that the frequency response is of this form. We will also derive a factorization of the system response for the different forms of symmetry.

2 Symmetry Conditions

2.1 Simplified Case

First consider a simplified case – we will add more complexity as we go along. The filter $G(z)$ will be causal and linear-phase with a non-zero first coefficient $g[0]$ and a non-zero last coefficient $g[N-1]$,

$$G(z) = \sum_{n=0}^{N-1} g[n] z^{-n}. \tag{2}$$

Consider the frequency response corresponding to $G(z)$,

$$G(\omega) = e^{-j\alpha\omega} \, B(\omega), \tag{3}$$

where $B(\omega)$ is purely real. We can also consider the backwards relationship corresponding to Eq. (3),

$$B(\omega) = e^{j\alpha\omega} G(\omega). \tag{4}$$

In this form, we are trying to find the requirements on the coefficients of $G(\omega)$ such that $B(\omega)$ becomes purely real.

To accommodate both even and odd numbers of coefficients, we will consider the up-sampled response[1]

$$B(z^2) = z^{2\alpha} G(z^2). \tag{5}$$

The upsampling operation which forms $G(z^2)$ from $G(z)$, inserts zero-valued samples between the coefficients, giving an odd-length sequence of length $2N-1$. The samples of the response corresponding to $G(z^2)$ are

$$g_2[n] = \begin{cases} g[n/2], & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{6}$$

2.2 Integer Delay

First consider the case that 2α is an integer. Then $b_2[n]$ is just a shifted version of $g_2[n]$,

$$b_2[n] = g_2[n + 2\alpha]. \tag{7}$$

For a frequency response $X(\omega)$ to be real-valued, the coefficients must satisfy $x[n] = x^*[-n]$. Identifying $B(z^2)$ with $X(z)$, then for the frequency response $B(2\omega)$ to be real, the coefficients must have conjugate symmetry,

$$b_2[n] = b_2^*[-n]. \tag{8}$$

Given that $b_2[n]$ is just a shifted version of $g_2[n]$, and that $g_2[n]$ is of finite length, the shift must centre $g_2[n]$ around zero. The delay α is then an integer or half-integer,

$$\alpha = \frac{N-1}{2}. \tag{9}$$

The upsampled coefficients $b_2[n]$ will have $N-1$ samples to the left of zero and $N-1$ to the right of zero.

We can express $g[n]$ in terms of the non-zero coefficients of $b_2[n]$,

$$g[n] = b_2[2n - (N-1)], \qquad \text{for } n = 0, \ldots, N-1. \tag{10}$$

Using the conjugate symmetry of $b_2[n]$,

$$g[n] = g^*[N-1-n]. \tag{11}$$

In terms of z-transforms,

$$G(z) = z^{-(N-1)} G^*(1/z^*). \tag{12}$$

[1] This artifice is being introduced to solve that problem that for an even number of coefficients, we cannot shift the FIR time response corresponding to $G(z)$ to be centred at zero.

The double conjugation (once on z and again on $G(\cdot)$), leaves the z value unconjugated, and the coefficients conjugated. The frequency response in Eq. (3) becomes

$$G(\omega) = e^{-j\omega(N-1)/2} B(\omega), \tag{13}$$

where $B(\omega)$ is real.

2.3 General Delay

For 2α non-integer, we have to consider an interpolation operation. First form a bandlimited continuous-time signal from $g[n]$,

$$g(t) = \sum_{n=0}^{N-1} g[n] \operatorname{sinc}(t-n), \tag{14}$$

where $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$. The digital signal $g[n]$ is linear-phase if and only if [1]

$$g(t+\alpha) = g^*(\alpha - t). \tag{15}$$

These terms can be expanded as follows,

$$\begin{aligned} g(t+\alpha) &= \sum_{n=0}^{N-1} g[n] \operatorname{sinc}(t+\alpha-n), \\ g^*(\alpha-t) &= \sum_{n=0}^{N-1} g^*[N-1-n] \operatorname{sinc}(t+N-1-\alpha-n). \end{aligned} \tag{16}$$

If we set $g[n] = g^*[N-1-n]$, then conjugate symmetry forces $\alpha = (N-1)/2$. If we set $\alpha = (N-1)/2$, then $g[n] = g^*[N-1-n]$. No other value of α together with a finite number of coefficients, will allow $g(t)$ to satisfy the conjugate symmetry requirements.

Perhaps a bit surprising is the result that there do exist bandlimited, continuous-time signals which when sampled give a *causal* infinite-extent (IIR) linear-phase digital signal [1]. However, the resulting IIR response $g[n]$ does *not* correspond to a rational z-transform.

2.4 Constant Phase Term

For more generality, we can add a constant phase term to Eq. (3),

$$G(\omega) = e^{-j\alpha\omega} e^{j\beta} B(\omega). \tag{17}$$

Then setting $\alpha = (N-1)/2$,

$$B(z^2) = e^{-j\beta} z^{N-1} G(z^2). \tag{18}$$

The relationship in terms of filter coefficients is

$$b_2[2n - (N-1)] = e^{-j\beta} g[n]. \tag{19}$$

Then using the conjugate symmetry of $b_2[n]$,

$$g[n] = e^{j2\beta} g^*[N-1-n]. \tag{20}$$

or in z-transform notation,

$$G(z) = z^{-(N-1)} e^{j2\beta} G^*(1/z^*). \tag{21}$$

There is another way of expressing the coefficient symmetries,

$$g[n] = e^{-j\beta} \tilde{g}[n], \qquad \text{where} \quad \tilde{g}[n] = \tilde{g}^*[N-1-n]. \tag{22}$$

With this view, we can see that the filter with coefficients $g[n]$ has the same zeros as the filter with coefficients $\tilde{g}[n]$.

If the coefficients are real, we see from Eq. (20) that $e^{j2\beta}$ must be real, i.e. β must be 0, $\pm\pi$, or $\pm\pi/2$.[2]

2.5 Symmetry Imposed by the Phase Term

The symmetry constraints imposed by Eq. (20) are shown schematically in Table 1. The rows of that table are summarized as follows.

1. For a general β and complex coefficients,

   $$g[N-1-n] = e^{j2\beta} g^*[n]. \tag{23}$$

   If the number of coefficients is odd, the middle coefficient must be of the form $re^{j\beta}$, where r is real.

2. If $\beta = 0$ or $\beta = \pm\pi$, the coefficients have conjugate symmetry,

   $$g[N-1-n] = g^*[n]. \tag{24}$$

[2] For considerations of the coefficient symmetry, only two values of β need be distinguished: 0 and $\pi/2$.

If the number of complex coefficients is odd, there is a middle coefficient that must be its own conjugate, i.e. the middle coefficient must be real.

3. For complex coefficients, if $\beta = \pm\pi/2$, the coefficients must be negative conjugates of each other,

$$g[N-1-n] = -g^*[n]. \tag{25}$$

If the number of coefficients is odd, the middle coefficient must be purely imaginary or zero.

4. For real coefficients, the phase β can only be 0, $\pm\pi$, or $\pm\pi/2$. For $\beta = 0$ or $\beta = \pm\pi$, the (real) coefficients have an even symmetry,

$$g[N-1-n] = g[n]. \tag{26}$$

5. For $\beta = \pm\pi/2$, the (real) coefficients have an odd symmetry,

$$g[N-1-n] = -g[n]. \tag{27}$$

If N is odd, the middle coefficient is zero.

Table 1: Coefficient symmetries for linear-phase FIR filters. The first 3 rows show the configurations for complex coefficients. The last 2 rows show the configurations for real coefficients. In the table u and v are complex; a, b, c, and r are real.

β	N odd	N even
general	$\begin{bmatrix} u & v & re^{j\beta} & v^* e^{j2\beta} & u^* e^{j2\beta} \end{bmatrix}$	$\begin{bmatrix} u & v & v^* e^{j2\beta} & u^* e^{j2\beta} \end{bmatrix}$
0 or $\pm\pi$	$\begin{bmatrix} u & v & r & v^* & u^* \end{bmatrix}$	$\begin{bmatrix} u & v & v^* & u^* \end{bmatrix}$
$\pm\pi/2$	$\begin{bmatrix} u & v & jr & -v^* & -u^* \end{bmatrix}$	$\begin{bmatrix} u & v & -v^* & -u^* \end{bmatrix}$
0 or $\pm\pi$	$\begin{bmatrix} a & b & c & b & a \end{bmatrix}$	$\begin{bmatrix} a & b & b & a \end{bmatrix}$
$\pm\pi/2$	$\begin{bmatrix} a & b & 0 & -b & -a \end{bmatrix}$	$\begin{bmatrix} a & b & -b & -a \end{bmatrix}$

2.6 Shifted Filter Coefficients

The final generality will be a shift of K samples,

$$H(z) = z^{-K} G(z). \tag{28}$$

The first non-zero coefficient is now at time $n = K$, and the last non-zero coefficient is at time $n = K + N - 1$. The symmetry conditions are

$$h[2K + N - 1 - n] = e^{j2\beta} h^*[n]. \tag{29}$$

In terms of the transfer function this is

$$H(z) = z^{-(2K+N-1)} e^{j2\beta} H^*(1/z^*). \tag{30}$$

The filter $H(z)$ will have the same singularities in the finite z-plane ($0 < |z| < \infty$) as $G(z)$, but will modify the number of singularities at $z = 0$ and/or $z = \infty$ due to the shift operation.

3 Zero Symmetries of the Filter Response

Consider the singularities of $G(z)$. This causal filter has $N - 1$ poles at the origin, and $N - 1$ zeros in the finite z-plane .

3.1 Zero Symmetries: Complex Coefficients

From Eq. (21), we see that if z_k is a zero of $G(z)$, then its conjugate-reciprocal $1/z_k^*$ is also a zero. Write z_k as $z_k = r_k e^{j\theta_k}$. Then the conjugagte-reciprocal zero is $1/z_k^* = (1/r_k)e^{j\theta_k}$. Zeros on the unit circle are their own conjugate-reciprocals, and hence can appear singly. The zero symmetries for complex coefficients are illustrated in Fig. 1.

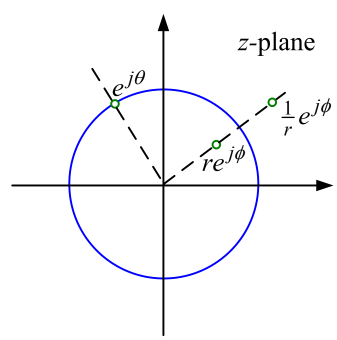

Fig. 1: Zero symmetries for complex coefficients

For an odd number of coefficients N, there are an even number of zeros. Since zeros off the unit circle occur in conjugate-reciprocal pairs, the number of zeros on the unit circle must be even.

For an even number of coefficients, there must be an odd number of zeros on the the unit circle, i.e. there must be at least one zero on the unit circle.

3.2 Zero Symmetries: Real Coefficients

For real coefficients, the zeros of $G(z)$ must appear in conjugate-reciprocal pairs *and* complex conjugate pairs. If $z_k = r_k e^{j\theta_k}$ is a zero of $G(z)$, then so are $z_k^* = r_k e^{-j\theta_k}$, $1/z_k^* = (1/r_k)e^{j\theta_k}$, and $1/z_k = (1/r_k)e^{-j\theta_k}$. Complex zeros off the unit circle appear in fours. Complex zeros on the unit circle appear in pairs. Real zeros off the unit circle also appear in pairs. Real zeros at $z = \pm 1$ can appear singly. The zero symmetries for real coefficients are shown in Fig. 2.

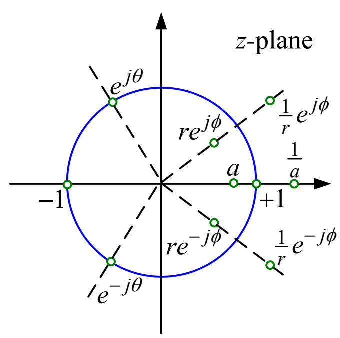

Fig. 2: Zero symmetries for real coefficients

4 Constrained Zeros: Real Coefficients

For real coefficients, some of the zeros are constrained to appear at $z = \pm 1$. We have already seen that for real coefficients, the constant phase term β can only take on values of 0, $\pm\pi$, or $\pm\pi/2$. There are four cases to consider, designated as linear-phase filters of types I through IV, see Table 2. These are the cases shown in the last two rows of Table 1.

Specializing Eq. (21) for real coefficients, we can write

$$G(z) = \frac{1}{2}\left[G(z) + z^{-(N-1)} e^{j2\beta} G(1/z)\right]. \tag{31}$$

We will examine this relationship for $z = \pm 1$.

Table 2: Filter type designations for linear-phase FIR filters with real coefficients

Type	β	Symmetry	N
I	0 or $\pm\pi$	$g[n] = g[N-1-n]$	Odd
II	0 or $\pm\pi$	$g[n] = g[N-1-n]$	Even
III	$\pm\pi/2$	$g[n] = -g[N-1-n]$	Odd
IV	$\pm\pi/2$	$g[n] = -g[N-1-n]$	Even

4.1 Zeros at $z = 1$

For $z = 1$,

$$G(1) = \frac{G(1)}{2}\left[1 + e^{j2\beta}\right]. \tag{32}$$

The response will be zero at $z = 1$ if β is $\pm\pi/2$. Then $G(z)$ has a zero at $z = 1$ for filter types III and IV.

4.2 Zeros at $z = -1$

For $z = -1$,

$$G(-1) = \frac{G(-1)}{2}\left[1 + (-1)^{-(N-1)}e^{j2\beta}\right]. \tag{33}$$

The response will be zero at $z = -1$ if N is even and β is 0 or $\pm\pi$, or if N is odd and β is $\pm\pi/2$. Then $G(z)$ has a zero at $z = -1$ for filter types II and III.

4.3 Fixed Root Factors

Noting the fixed roots found above, we can separate out a factor containing the fixed roots to give

$$G(z) = Q(z)P(z), \tag{34}$$

where $Q(z)$ has roots only at $z = \pm 1$. The filter $P(z)$, as will be shown shortly, is even-symmetric with an odd number of coefficients. The results are summarized in the Table 3.

If we designate the number of coefficients in $P(z)$ as M, then the number of coefficients in $Q(z)$ is $N - M$. The symmetry of $Q(z)$ can be expressed as

$$Q(z) = \pm z^{-(N-M)}Q(1/z), \tag{35}$$

where the upper sign applies for $G(z)$ being even-symmetric (Types I and II), and the lower sign

Table 3: Fixed filter factors for linear-phase FIR filters with real coefficients

Type	Description	$Q(z)$	No. Coef. $P(z)$
I	N odd even-symmetric	1	N
II	N even even-symmetric	$1+z^{-1}$	$N-1$
III	N odd odd-symmetric	$1-z^{-2}$	$N-2$
IV	N even odd-symmetric	$1-z^{-1}$	$N-1$

applies for $G(z)$ being odd-symmetric (Types III and IV). Then

$$\begin{aligned} P(z) &= \frac{G(z)}{Q(z)} \\ &= \frac{\pm z^{-(N-1)} Q(1/z) P(1/z)}{Q(z)} \\ &= \frac{\pm z^{-(N-1)} [\pm z^{N-M} Q(z)] P(1/z)}{Q(z)} \\ &= z^{-(M-1)} P(1/z). \end{aligned} \tag{36}$$

In this equation, there are only two cases: $G(z)$ even-symmetric (use the plus signs) and $G(z)$ odd-symmetric (use the minus signs). Then for all filter types, $P(z)$ is a Type I filter (odd number of coefficients and even-symmetric).

4.4 Frequency Response

We can write the frequency response of $G(z)$ as

$$\begin{aligned} G(\omega) &= Q(\omega)P(\omega) \\ &= e^{-j\omega(N-M)/2} e^{j\phi} Q_0(\omega) e^{-j\omega(M-1)/2} P_0(\omega) \\ &= e^{-j\omega(N-1)/2} e^{j\phi} Q_0(\omega) P_0(\omega). \end{aligned} \tag{37}$$

We have expressed the frequency response in terms of so-called zero-phase responses[3] $Q_0(\omega)$ and $P_0(\omega)$. The zero-phase response $P_0(\omega)$ can be written as

$$\begin{aligned} P_0(\omega) &= \sum_{n=-(M-1)/2}^{(M-1)/2} p[n] e^{-j\omega n} \\ &= p[0] + 2 \sum_{n=1}^{(M-1)/2} p[n] \cos(\omega n). \end{aligned} \tag{38}$$

This shows that $P_0(\omega)$ is real-valued.

For each $Q(z)$ in Table 3, we can find the corresponding value of ϕ and $Q_0(\omega)$. These are shown in Table 4.

Table 4: Zero-phase fixed factors for linear-phase FIR filters with real coefficients

Type	Description	ϕ	$Q_0(\omega)$
I	N odd even-symmetric	0	1
II	N even even-symmetric	0	$2\cos(\omega/2)$
III	N odd odd-symmetric	$\pi/2$	$2\sin(\omega)$
IV	N even odd-symmetric	$\pi/2$	$2\sin(\omega/2)$

4.5 Notes on the Factors of the Frequency Response

1. The $Q_0(\omega)$ term captures at most one zero at $z = 1$ and one zero at $z = -1$. The $P_0(\omega)$ term will include the effect of the remaining zeros at $z = \pm 1$, which will be of even multiplicity.

2. Since the frequency response of filters of Type II or Type III have a null at $\omega = \pi$ due to $Q(\omega)$, they are unsuitable for the implementation of highpass filters. Since the frequency response of filters of Type III or Type IV have a null at dc due to $Q(\omega)$, they are unsuitable for the implementation of lowpass filters.

3. The zero-phase factor $P_0(\omega)$ in Eq. (38) is periodic in ω with the requisite period of 2π. However, for even filter lengths (filter Types II and IV), the zero-phase factor $Q_0(\omega)$ in Table 4 is periodic with period 4π. It is the linear-phase term in Eq. (37) which will ensure that the

[3] Zero-phase responses are real responses which contribute a frequency dependent phase of 0 or $\pm\pi$.

overall response has period 2π. Designate the linear-phase term as $D(\omega) = e^{-j\omega(N-1)/2}$. For N even,

$$\begin{aligned} D(\omega + 2\pi) &= -D(\omega), \\ Q_0(\omega + 2\pi) &= -Q_0(\omega). \end{aligned} \tag{39}$$

The product $D(\omega)Q_0(\omega)$ is periodic, with the "proper" period 2π.

5 Summary

We have shown that the frequency response of a generalized linear-phase filter with real coefficients can be written as the product of several factors,

$$G(\omega) = e^{-j\omega(N-1)/2} e^{j\beta} Q_0(\omega) P_0(\omega). \tag{40}$$

For real filters, $Q_0(\omega)$ is a real response which depends *only* on whether the filter is of even or odd length and whether the filter is symmetric or anti-symmetric. The factor $P_0(\omega)$ is a real.

For complex coefficients,

$$G(\omega) = e^{-j\omega(N-1)/2} e^{j\beta} B(\omega), \tag{41}$$

where $B(\omega)$ is real.

The coefficients of the generalized linear-phase filter obey

$$g[N-1-n] = e^{j2\beta} g^*[n]. \tag{42}$$

For real coefficients, β is restricted to be 0, $\pm\pi$, or $\pm\pi/2$.

References

[1] M. A. Clements and J. W. Pease, “On Causal Linear Phase IIR Filters”, *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. 37, no. 4, pp. 479–484, April 1989.