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Frequency Domain Representations of Sampled and Wrapped Signals

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- 2011-03 v1.2: Creative Commons licence, minor updates (coloured equations)
- 2009-09 v1c: Updated
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1 Introduction

These notes examine the relationships between frequency domain representations of discrete-time and wrapped signals derived from a continuous-time signal. The first part of these notes develops the relationships for periodic signals which allow for the analysis of periodic signals within the framework of the Fourier transform. With this formalism, it is shown that sampling in one domain (time or frequency) corresponds to wrapping (aliasing) in the other domain (frequency or time).

The second part examines the relationships between the Fourier series, the Discrete-Time Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT).

Throughout this document, round brackets are used for functions of continuous variables (examples: $v(t)$ and $V(\omega)$); square brackets are used for functions of discrete variables (example: $v[n]$). In the first part of this document, the equations shown within boxes summarize results that are used in the developments leading to formulations for the Fourier transform of periodic signals. In the second part of this document, the equations shown within boxes are results which appear on the diagram relating the Fourier domain representations of sampled and wrapped signals.

2 Continuous-Time Fourier Transform

The Fourier transform of a continuous-time signal is given by

$$V(F) = \int_{-\infty}^{\infty} v(t) e^{-j2\pi Ft}\, dt. \tag{1}$$

This is well-defined if $v(t)$ is absolutely integrable and has a finite number of extrema and finite discontinuities, in a finite interval [1]. The inverse Fourier transform is

$$v(t) = \int_{-\infty}^{\infty} V(F) e^{j2\pi Ft}\, dF. \tag{2}$$

Periodic signals, which are non-decaying and hence not absolutely integrable, are amenable to a Fourier series expansion. The Fourier transform can be associated with the Fourier series expansion by using the Dirac delta function.

The forward and inverse transforms differ only in the sign of the exponent. This gives a duality relationship:

$$\text{If } v(t) \Longleftrightarrow V(F), \text{ then } V(t) \Longleftrightarrow v(-F). \tag{3}$$

2.1 Dirac Delta Function

The Dirac delta (impulse function) can be defined in terms of its properties [2]. The sampling property of the delta function (more properly a distribution) is

$$\int_{t \in T_A} v(t)\delta(t)\,dt = \begin{cases} v(0), & 0 \in T_A, \\ 0, & 0 \notin T_A. \end{cases} \tag{4}$$

From this characterization, the delta function can be shown to be zero everywhere except at the origin, yet it has unit area,

$$\begin{aligned} \delta(t) &= 0, \qquad \text{for } t \neq 0, \\ \int_{-\infty}^{\infty} \delta(t)\,dt &= 1. \end{aligned} \tag{5}$$

The formal operations involving the delta function in an integral result in

$$\int_{-\infty}^{\infty} v(t - t_o)\delta(t)\,dt = \int_{-\infty}^{\infty} v(t)\delta(t + t_o)\,dt \tag{6}$$

and

$$\int_{-\infty}^{\infty} v(at)\delta(t)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} v(t)\delta(t/a)\,dt. \tag{7}$$

2.1.1 Fourier transform: Delta function

Using the sampling property of the delta function, the Fourier transform of the delta function evaluates to a constant,

$$\int_{-\infty}^{\infty} \delta(t) e^{-j2\pi Ft}\,dt = 1. \tag{8}$$

The inverse Fourier transform gives

$$\int_{-\infty}^{\infty} e^{j2\pi Ft}\,dF = \delta(t). \tag{9}$$

This integral must be evaluated using the Cauchy principal value, i.e., as the limit

$$\lim_{T \to \infty} \int_{-T/2}^{T/2} e^{j2\pi Ft}\,dF. \tag{10}$$

Note that since $\delta(t)$ behaves like a symmetric function, the exponent in the integral can have either sign.

The inverse transform giving a delta function in Eq. (9) gives a formula for the integral of a complex exponential. That result can be restated using symbols which do not evoke time or frequency,

$$\boxed{\int_{-\infty}^{\infty} e^{\pm j2\pi ux}\, du = \delta(x).} \tag{11}$$

2.2 Fourier Series: Continuous-Time Signal

A continuous-time periodic function (subject to the Dirichlet conditions: absolute integrability over a period, a finite number of extrema and finite discontinuities, in a finite interval [1]) has a Fourier series expansion in complex exponentials. Consider a periodic function $\tilde{v}(t)$ with period T. The Fourier series expansion for $\tilde{v}(t)$ is

$$\tilde{v}(t) = \sum_{m=-\infty}^{\infty} v_m e^{j2\pi mt/T}. \tag{12}$$

The Fourier series coefficients are found from

$$v_m = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{v}(t) e^{-j2\pi mt/T}\, dt. \tag{13}$$

2.3 Fourier Transform: Continuous-Time Periodic Signal

Express the Fourier transform of $\tilde{v}(t)$ in terms of the Fourier series coefficients to get

$$\begin{aligned} V_p(F) &= \int_{-\infty}^{\infty} \tilde{v}(t) e^{-j2\pi Ft}\, dt \\ &= \sum_{m=-\infty}^{\infty} v_m \int_{-\infty}^{\infty} e^{-j2\pi t(F-m/T)}\, dt \\ &= \sum_{m=-\infty}^{\infty} v_m \delta\big(F - \frac{m}{T}\big). \end{aligned} \tag{14}$$

Equation (11) has been used to evaluate the integral of the complex exponential. Summarizing, the Fourier transform of a periodic function is a sequence of delta functions in the frequency domain at the harmonics of the periodic signal repetition rate. The areas of the delta functions are given by the Fourier series coefficients,

$$\boxed{V_p(F) = \sum_{m=-\infty}^{\infty} v_m \delta\big(F - \frac{m}{T}\big).} \tag{15}$$

2.3.1 Fourier transform: Periodic impulse train

Consider the periodic impulse train (period T),

$$\tilde{v}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{16}$$

The Fourier series coefficients for this signal are given by

$$v_m = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t - \frac{k}{T}) e^{j2\pi mt/T} \, dt. \tag{17}$$

The only delta function within the integration range corresponds to $k = 0$. Using the sampling property of the delta function, the integral evaluates to unity. Then the Fourier series coefficients are constant ($v_m = 1/T$) and the Fourier transform of the impulse train is

$$V_p(F) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(F - \frac{m}{T}). \tag{18}$$

Periodic functions have delta functions in their Fourier transforms and delta functions have periodic functions in their Fourier transforms. The duality between the forward and inverse Fourier transforms (Eq. (3)) shows that an impulse train (periodic with delta functions) must have as its Fourier transform another impulse train (delta functions and periodic).

An alternate formulation for the Fourier transform (or inverse Fourier transform) of an impulse train can be derived. The Fourier transform for a delayed delta function $\delta(t - kT)$ is $e^{-j2\pi kTF}$. Then the Fourier transform pair is

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \Longleftrightarrow \sum_{k=-\infty}^{\infty} e^{-j2\pi kTF}. \tag{19}$$

Using the time-frequency duality of the Fourier transform,

$$\boxed{\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} e^{j2\pi mt/T} \Longleftrightarrow \sum_{k=-\infty}^{\infty} e^{-j2\pi kFT} = \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(F - \frac{m}{T}).} \tag{20}$$

2.4 Periodic Wrapped Continuous-Time Signals

Consider forming a periodic signal $\tilde{v}(t)$ from a (non-periodic) signal $v(t)$ as follows

$$\tilde{v}(t) = v(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} v(t - kT). \tag{21}$$

The first part of this expression is an "operational" description of the process of forming a periodic signal as the convolution with an impulse train. The second part gives an interpretation in terms of *wrapping* the time-domain signal.[1] Using the fact that a convolution in the time-domain corresponds to a product in the frequency domain (see Appendix A), the Fourier transform of $\tilde{v}(t)$ is

$$\sum_{k=-\infty}^{\infty} v(t-kT) \Longleftrightarrow V(F)\,\frac{1}{T}\sum_{m=-\infty}^{\infty}\delta(F-\frac{m}{T}) = \frac{1}{T}\sum_{m=-\infty}^{\infty} V(\frac{m}{T})\delta(F-\frac{m}{T}). \qquad (22)$$

The Fourier series coefficients are given by $V(F)/T$ evaluated (sampled) at the harmonic frequencies. Here $V(F)$ is the Fourier transform of $v(t)$, where $v(t)$ can be longer than T. This relationship shows the wrapped time domain signal corresponds to a sampled frequency domain response.

The Fourier series coefficients can be found directly from Eq. (13). Noting that Eq. (13) is just a scaled version of the Fourier transform,

$$v_m = \frac{1}{T} V_T(\frac{m}{T}), \qquad (23)$$

where $V_T(m/T)$ is a sample of the Fourier transform of one period of $\tilde{v}(t)$.

Given a signal $v(t)$ which is wrapped to become $\tilde{v}(t)$, there are two ways to get the coefficients defining the frequency response of $\tilde{v}(t)$. The first is to take the Fourier transform of $v(t)$ (which can be longer than T) and then sample the frequency response at $F = m/T$. The second is to take the Fourier transform of one period of $\tilde{v}(t)$ and then sample the frequency response at $F = m/T$.

2.4.1 Poisson sum formula

Using Eq. (22), take the term-by-term inverse Fourier transform of the extreme righthand side expression and equate it to the lefthand side. This gives the Poisson sum formula,

$$\boxed{\sum_{k=-\infty}^{\infty} v(t-kT) = \frac{1}{T}\sum_{m=-\infty}^{\infty} V(\frac{m}{T}) e^{j2\pi tm/T}.} \qquad (24)$$

This formula gives the Fourier series expansion of the wrapped signal.

3 Sampling

The process of sampling a continuous-time signal can be modelled as the multiplication of the continuous-time signal by a impulse train sampling function. The areas of the resulting impulses

[1] The continuous-time signal is wrapped onto a circle of circumference T with superimposed intervals being added.

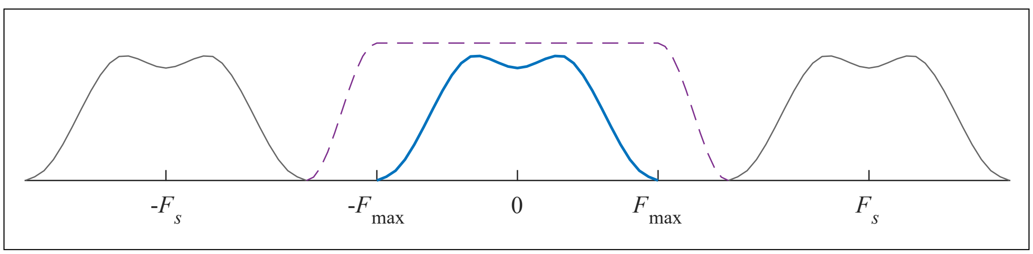
are the sample values,

$$v_s(t) = v(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} v(kT)\delta(t - kT). \tag{25}$$

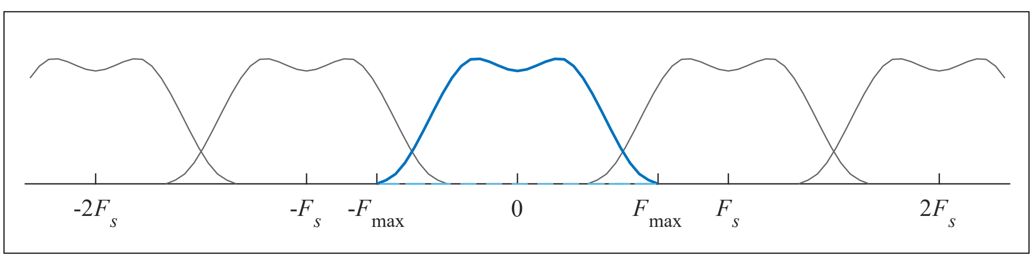
The Fourier transform $V_s(F)$ can be computed using the relationship that a product in the time domain corresponds to a convolution in the frequency domain,

$$v_s(t) = v(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) \Longleftrightarrow V_s(F) = V(F) * \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(F - \frac{m}{T}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} V(F - \frac{m}{T}). \tag{26}$$

The first part of the frequency-domain expression is the operational form for calculating the frequency response. This frequency response consists of wrapped versions of $V(F)$. This relationship shows that sampling in the time domain corresponds to wrapping in the frequency domain. The resulting frequency response will not have aliasing (overlapping responses) if the baseband signal $V(F)$ is bandlimited to $|F| < F_s/2$. Figure 1 shows the spectrum $V_s(F)$ for sampling without and with aliasing.



(a) Sampling with no aliasing, $F_s > 2F_{\max}$



(b) Sampling with aliasing, $F_s < 2F_{\max}$

Fig. 1 Spectrum after sampling, sampling rate $F_s = 1/T$. The upper plot shows the case with no aliasing, while the lower plot shows the case with aliasing. The dashed spectrum in the upper plot shows the response of an interpolating filter for reconstructing the continuous-time signal from its samples.

There is another expression for the frequency response of the sampled signal in Eq. (28), This time take the Fourier transform term-by-term of the second form of the time-domain expression in Eq. (25) to give

$$v_s(t) = \sum_{k=-\infty}^{\infty} v(kT)\delta(t-kT) \Longleftrightarrow V_s(F) = \sum_{k=-\infty}^{\infty} v(kT)e^{-j2\pi kFT}. \tag{27}$$

Summarizing,

$$\boxed{v_s(t) = \sum_{k=-\infty}^{\infty} v(kT)\delta(t-kT) \Longleftrightarrow V_s(F) = \frac{1}{T}\sum_{m=-\infty}^{\infty} V\big(F - \frac{m}{T}\big) = \sum_{k=-\infty}^{\infty} v(kT)e^{-j2\pi kFT}.} \tag{28}$$

The sample values $v(nT)$ are the discrete-time sequence $x[n]$ for the Discrete Fourier Transform of §4.

3.1 Reconstruction of a Signal from its Samples

For a signal bandlimited to $|F| < F_{\text{max}}$, if the sampling rate $F_s = 1/T$ is large enough, there is no overlap of the shifted spectra in the representation of $V_s(F)$. The no-overlap condition is

$$F_s \geq 2F_{\text{max}}. \tag{29}$$

A lowpass filter that that rejects the shifted spectra can be used to reconstruct $x(t)$. The lowpass filter response is

$$H(F) = \begin{cases} 1, & |F| \leq F_{\text{max}}, \\ 0, & |F| > F_s - F_{\text{max}}. \end{cases} \tag{30}$$

Note that $H(F)$ is undefined in the transition between the passband and stopband. With no aliasing, the original signal can be reconstructed as

$$\begin{aligned} v(t) &= v_s(t) * h(t) \\ &= \sum_{k=-\infty}^{\infty} v(nT)h(t-kT). \end{aligned} \tag{31}$$

The lowpass filter interpolates between the sample values. If $F_s = 2F_{\text{max}}$, the transition band is zero width and an ideal lowpass filter is the only choice for interpolating the sample values,

$$H_0(F) = \begin{cases} 1, & |F| \leq F_s/2, \\ 0, & \text{elsewhere}. \end{cases} \tag{32}$$

The impulse response of this filter is the sinc function,

$$h_0(t) = \text{sinc}(t) = \frac{\sin(\pi t/T)}{\pi t/T}. \tag{33}$$

The sinc function has the zero-crossing property,[2]

$$h_0(kT) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases} \tag{34}$$

The Fourier transform relationship between the sinc function and the ideal lowpass filter response has to be interpreted in a mean-square sense. This is due to the fact that the sinc function is not absolutely integrable since its envelope decreases only as $1/|t|$. Truncating the sinc function leads to the Gibbs phenomenon [1, §4.2].

The absolute integrability requirement can be satisfied if $v(t)$ is oversampled, i.e., $F_s > 2F_{\text{max}}$. This allows for an interpolating function $h(t)$ which has a Fourier transform without discontinuities.

Start with the ideal lowpass filter response $H_0(F)$. Convolve this with a symmetric response $C(F)$ with unit area and which is zero outside of the interval $|F| \leq F_s/2 - F_{\text{max}}$,

$$\begin{aligned} H(F) &= C(F) * H_0(F) \\ h(t) &= c(t)h_0(t). \end{aligned} \tag{35}$$

Note that $h(t)$ retains the zero-crossing property of the sinc function. Choosing the support of $C(F)$ to be $|F| \leq F_s - F_{\text{max}}$ results in the largest transition band.

3.1.1 Example: Raised-cosine filter

An example of a suitable function $C(F)$ is

$$C(F) = \begin{cases} \dfrac{\pi}{2\alpha F_s} \cos\left(\dfrac{\pi F}{\alpha F_s}\right), & |F| \leq \alpha F_s/2, \\ 0, & \text{elsewhere}, \end{cases} \tag{36}$$

[2] A filter with the zero-crossing property is referred to as a Nyquist filter. In the frequency domain, a Nyquist filter has a constant aliased response.

where the parameter α is the fractional excess bandwidth over a minimum bandwidth response,

$$\alpha = 1 - \frac{F_{\max}}{F_s/2}. \tag{37}$$

The lowpass filter has a raised-cosine spectrum,

$$H(F) = \begin{cases} 1, & |F| \leq (1-\alpha)\frac{F_s}{2}, \\ \frac{1}{2}\big[1 - \sin\big(\frac{\pi}{\alpha}\frac{F_s}{2}(|F| - \frac{F_s}{2})\big)\big], & (1-\alpha)\frac{F_s}{2} < |F| < (1+\alpha)\frac{F_s}{2}, \\ 0, & \text{elsewhere.} \end{cases} \tag{38}$$

The raised-cosine spectrum is shown as a dashed line superimposed on the top plot in Fig. 1. The corresponding interpolation function is

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} \frac{\cos(\alpha \pi t/T)}{1 - (2\alpha t/T)^2}. \tag{39}$$

In this form, it can be seen that the response falls off asymptotically as $1/|t|^3$.

4 Discrete-Time Fourier Transform

The discrete-time Fourier transform (DTFT) applies to discrete-time signals and is given by

$$V(\omega) = \sum_{n=-\infty}^{\infty} v[n] e^{-j\omega n}. \tag{40}$$

This sum converges if $v[n]$ is absolutely summable. The frequency response is periodic, with period 2π. In the previous sections, the Fourier transform response was denoted as $V(F)$. For the DTFT the same symbol is reused, this time using normalized radian frequency variable ω.

In terms of sampling a continuous-time signal in §3,

$$v[n] = v(nT) \text{ and } V(\omega) = V(F)\big|_{F=\omega/(2\pi T)}. \tag{41}$$

This sum in the definition of the DTFT can be considered to be a Fourier series expansion of the periodic signal $V(\omega)$. The inverse discrete-time Fourier transform is equivalent to the computation of the Fourier series coefficients,

$$v[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} V(\omega) e^{j\omega n} \, d\omega. \tag{42}$$

4.1 Fourier Transform: Discrete-Time Periodic Signal

Let $\tilde{v}[n]$ be periodic with period N,

$$\tilde{v}[n+N] = \tilde{v}[n]. \tag{43}$$

Evaluating the Fourier transform of this signal,

$$\begin{aligned} V_p(\omega) &= \sum_{n=-\infty}^{\infty} \tilde{v}[n] e^{-j\omega n} \\ &= \sum_{p=-\infty}^{\infty} \sum_{q=0}^{N-1} \tilde{v}[pN+q] e^{-j\omega(pN+q)} \\ &= \sum_{p=-\infty}^{\infty} e^{-j\omega pN} \sum_{q=0}^{N-1} \tilde{v}[q] e^{-j\omega q}. \end{aligned} \tag{44}$$

The second line of this equation is a result of substituting $n = pN + q$. The third line results from exploiting the periodicity of $\tilde{v}[n]$. The second factor of the result is the DTFT of one period of $\tilde{v}[n]$.

The first factor (sum of complex exponentials) in the equation above can be expressed in terms of an impulse train. The form of the sum is a little different than that encountered earlier. Appendix B recasts the earlier results in terms of radian frequency. Then from Eq. (89) with $T = N$,

$$\begin{aligned} \sum_{p=-\infty}^{\infty} e^{-j\omega pN} &= \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N}) \\ &= \frac{2\pi}{N} \sum_{k=0}^{N-1} \delta(\langle\omega\rangle_{2\pi} - \frac{2\pi k}{N}) \end{aligned} \tag{45}$$

The second line uses the fact that N terms of the sum appear in each interval of length 2π – the notation $\langle\omega\rangle_{2\pi}$ gives ω modulo 2π. Finally,

$$\boxed{V_p(\omega) = 2\pi \sum_{k=0}^{N-1} V_k \delta(\langle\omega\rangle_{2\pi} - \frac{2\pi k}{N}),} \tag{46}$$

where

$$\boxed{V_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{v}[n] e^{-j2\pi nk/N}.} \tag{47}$$

The coefficients V_k are the Fourier series coefficients for the discrete periodic sequence with period N. They are obtained as the DTFT of one period of $\tilde{v}[n]$, evaluated at $\omega = 2\pi k/N$. As will be seen later, these coefficients are the same as the discrete Fourier transform, except for a scale factor.

4.2 Fourier Series: Discrete-Time Signal

The Fourier series expansion for a discrete time signal can be obtained by taking the inverse transform of Eq. (46),

$$\begin{aligned} \tilde{v}[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} V(\omega) e^{j\omega n}\, d\omega \\ &= \sum_{k=-\infty}^{\infty} V_k \int_{-\epsilon}^{2\pi-\epsilon} \delta\big(\omega - \frac{2\pi k}{N}\big) e^{j\omega n}\, d\omega \\ &= \sum_{k=0}^{N-1} V_k e^{j2\pi kn/N}. \end{aligned} \tag{48}$$

In the second line, the limits of the integration have been shifted so that the delta functions for $k = 0$ to $k = N-1$ fall within the limits. This is possible because the integrand is periodic with period 2π. The result is a Fourier series expansion with Fourier series coefficients V_k given by Eq. (47),

$$\boxed{\tilde{v}[n] = \sum_{k=0}^{N-1} V_k e^{j2\pi kn/N}.} \tag{49}$$

The discrete-time periodic signal $\tilde{v}[n]$ can be interpolated to form a continuous-time periodic signal $\tilde{v}(t)$. The N samples of $\tilde{v}(t)$ are formed for $t = nT/N$, viz., $\tilde{v}(nT/N) = \tilde{v}[n]$. Then Eq. (49) with the substitution $n = Nt/T$ becomes

$$\tilde{v}(t) = \sum_{k=0}^{N-1} V_k e^{j2\pi kt/T}. \tag{50}$$

This is a periodic function formed from N complex sinusoids and can be compared with Eq. (12) which is a general periodic function with an unbounded number of terms. The Fourier series coefficients V_k are equal to a samples of the Fourier spectrum scaled by N as shown in Eq. (47).

4.3 Fourier Transform of a Discrete-Time Pulse Train

Consider the discrete-time pulse train

$$\tilde{v}[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]. \tag{51}$$

Here the delta function with square brackets is the unit pulse, equal to one if its argument is zero, and equal to zero otherwise. The Fourier series coefficients for this signal are constants at

$V_k = 1/N$. Then the Fourier series representation is

$$\sum_{k=-\infty}^{\infty} \delta[n-kN] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j2\pi nk/N}. \tag{52}$$

The DTFT of this pulse train can be found term-by-term for the lefthand side of the equation above,

$$V_p(\omega) = \sum_{k=-\infty}^{\infty} e^{-j\omega kN}. \tag{53}$$

An impulse train representation for this expression comes from Eq. (46). This gives the following representations of a discrete-time pulse train.

$$\boxed{\sum_{k=-\infty}^{\infty} \delta[n-kN] = \frac{1}{N}\sum_{m=0}^{N-1} e^{j2\pi nm/N} \Longleftrightarrow \sum_{k=-\infty}^{\infty} e^{-j\omega kN} = \frac{2\pi}{N}\sum_{m=0}^{N-1} \delta\big(\langle\omega\rangle_{2\pi} - \frac{2\pi m}{N}\big).} \tag{54}$$

This expression has a discrete-time pulse train on the left and a frequency-domain pulse train on the right.

4.4 Periodic Wrapped Discrete-Time Signals

Consider forming a periodic signal $\tilde{v}[n]$ from $v[n]$,

$$\tilde{v}[n] = v[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN] = \sum_{k=-\infty}^{\infty} v[n-kN]. \tag{55}$$

Using the fact that a convolution in the time-domain corresponds to a product in the frequency domain, the DTFT of $\tilde{v}[n]$ is

$$\sum_{k=-\infty}^{\infty} v[n-kN] \Longleftrightarrow V(\omega)\frac{2\pi}{N}\sum_{m=0}^{N-1} \delta\big(\langle\omega\rangle_{2\pi} - \frac{2\pi m}{N}\big) = \frac{2\pi}{N}\sum_{m=0}^{N-1} V\big(\frac{2\pi m}{N}\big)\delta\big(\langle\omega\rangle_{2\pi} - \frac{2\pi m}{N}\big). \tag{56}$$

Given a signal $v[n]$ which is wrapped to become $\tilde{v}[n]$, there are two ways to get the coefficients of the frequency response. The first is to take the Fourier transform of $v[n]$ and then sample the frequency response at $\omega = 2\pi m/N$. The second is to take the Fourier transform of one period of $\tilde{v}[n]$ and then sample the frequency response at $\omega = 2\pi m/N$.

4.4.1 Poisson sum formula

For discrete-time signals a result similar to the Poisson sum formula for continuous-time signals can be derived. In this case, taking the term-by-term inverse Fourier transform of the extreme right-hand side of the equation above,

$$\boxed{\sum_{k=-\infty}^{\infty} v[n-kN] = \frac{1}{N} \sum_{m=0}^{N-1} V\Big(\frac{2\pi m}{N}\Big) e^{j2\pi nm/N}.} \tag{57}$$

This formula gives the Fourier series expansion of the wrapped signal.

5 Discrete Fourier Transform

The discrete Fourier transform (DFT) for a sequence $x[n]$ is

$$V[k] = \sum_{n=0}^{N-1} v[n] e^{-j2\pi nk/N}. \tag{58}$$

The DFT coefficients $V[k]$ are periodic with period N. The inverse discrete Fourier transform is

$$v[n] = \frac{1}{N} \sum_{k=0}^{N-1} V[k] e^{j2\pi nk/N}. \tag{59}$$

In this equation, allowing n to take on any integer value, $v[n]$ becomes periodic with period N.

5.1 DTFT from DFT - Periodic Interpretation

The DFT can be considered to operate on one period of a periodic signal. With that interpretation, the DFT formula Eq. (58) differs only by a scale factor from the formula for calculating the discrete Fourier series coefficients in Eq. (48),

$$V[k] = NV_k, \qquad 0 \le k \le N-1. \tag{60}$$

From Eq. (46), the DTFT of the periodic signal can be expressed in terms of the DFT coefficients as

$$V_p(\omega) = \frac{2\pi}{N} \sum_{k=0}^{N-1} V[k] \delta\big(\langle\omega\rangle_{2\pi} - \frac{2\pi k}{N}\big). \tag{61}$$

The DTFT of the periodic input has discrete frequency components.

5.2 DTFT from DFT - Finite Length Signal

The DFT formula Eq. (58) can be applied to a finite length signal. Start with the periodic signal interpretation and use a window to extract one period to form the finite length sequence. In the time domain, the windowing operation is

$$v_N[n] = \tilde{v}_N[n]p_N[n], \tag{62}$$

where

$$p_N[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{elsewhere.} \end{cases} \tag{63}$$

In the frequency domain, the DTFT of $p_N[n]$ is

$$P_N(\omega) = N\Phi_N\omega, \tag{64}$$

where $\Phi_N(\omega)$ is the phase-shifted Dirichlet kernel (also known as the digital sinc function [1]),

$$\Phi_N(\omega) = e^{-j\omega(N-1)/2}\,\frac{1}{N}\frac{\sin(\omega N/2)}{\sin(\omega/2)}. \tag{65}$$

The DTFT for this interpretation of the DFT can be written as the convolution of $V_p(\omega)$ and $P_N(\omega)$,

$$\begin{aligned} V(\omega) &= V_p(\omega) * N\Phi_N(\omega) \\ &= \sum_{k=0}^{N-1} V[k]\Phi_N(\omega - \frac{2\pi k}{N}). \end{aligned} \tag{66}$$

This is a formula for the periodic interpolation of the DFT coefficients to form the DTFT of a sequence of length N.

6 Relationships Between the Frequency-Domain Representations

The results of the previous sections allow for the examination of the relationships between the frequency representations of continuous-time signals, sampled signals, and periodic signals. Figure 2 shows a schematic form of the relationships.[3] Consider the time-domain signals shown in the figure, starting with $x(t)$ at the top. On the left side of the diagram, the signal $x[n]$ is formed by sampling $x(t)$ with sampling interval T. The signal $\tilde{x}[n]$ is formed by wrapping $x[n]$ with period N.

On the right side of the diagram, the signal $\tilde{x}(t)$ is formed by wrapping $x(t)$ with period NT. Sampling $\tilde{x}(t)$ with period T closes the loop and gives $\tilde{x}[n]$. Thus sampling then wrapping (on

[3] A similar figure appears in [1, Chapter 4].

the left side) is the same as wrapping and then sampling (on the right side – with the proviso that wrapping period is N times the sampling interval T).

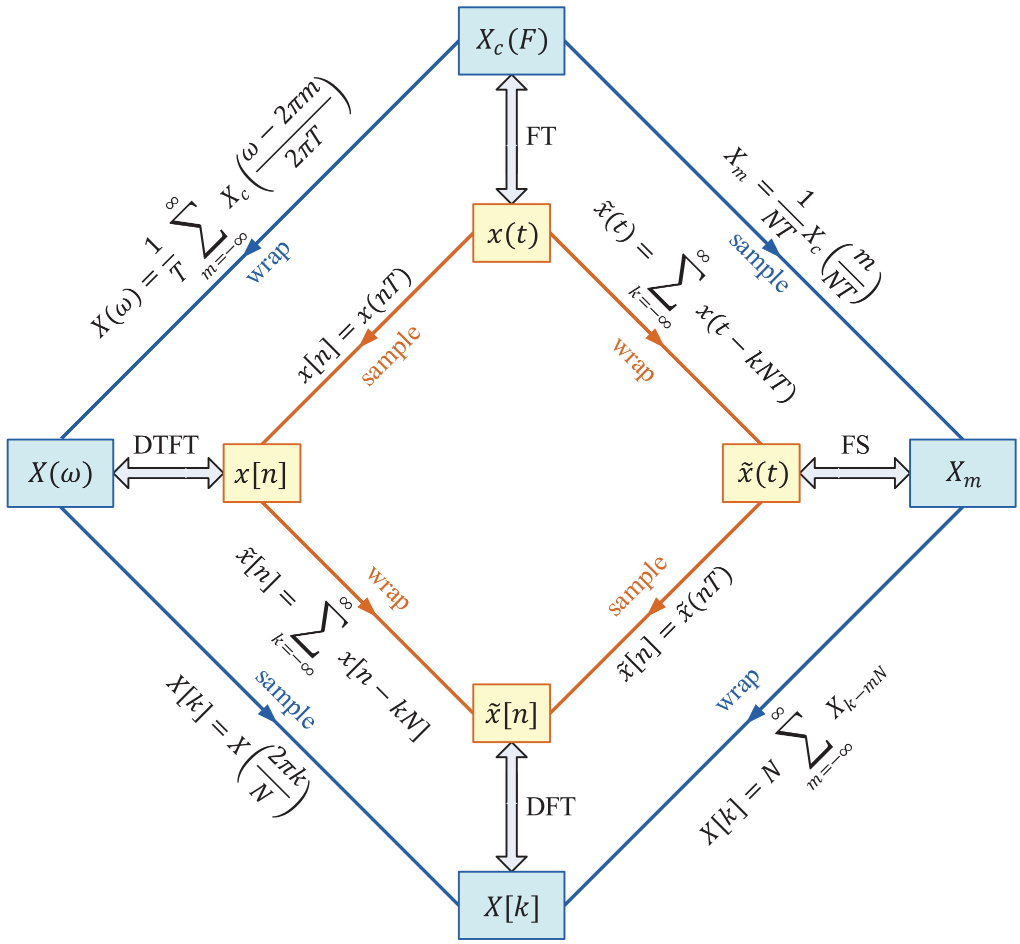


Fig. 2 Relationships between the frequency domain representations of continuous-time signals, sampled signals, and periodic signals. FT is the (continuous-time) Fourier transform, DTFT is the discrete-time Fourier transform, FS is the Fourier series, and DFT is the discrete Fourier transform.

6.1 Sampling a Continuous-Time Signal: $x(t) \to x[n]$

The process of sampling a continuous-time signal can be modelled as the multiplication of the continuous-time signal by a impulse train as shown in §3,

$$x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) \Longleftrightarrow X_c(F) * \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\big(F - \frac{m}{T}\big) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\big(F - \frac{n}{T}\big) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j2\pi nFT}. \tag{67}$$

This frequency response is periodic with period $1/T$ – sampling in the time domain corresponds to wrapping in the frequency domain. The resulting frequency response will not have aliasing (overlapping responses) if the baseband signal $X_c(F)$ is bandlimited to $|F| < 1/(2T)$.

In discrete-time,

$$x[n] = x(nT). \tag{68}$$

The DTFT of $x[n]$ is $X(\omega)$ which is periodic with period 2π. If the definition for the DTFT (Eq. (40)) is compared with the last term in Eq. (67),

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \tag{69}$$

with the mapping between F for the continuous-time Fourier transform and ω for the discrete-time Fourier transform being $\omega = 2\pi FT$. With this mapping, when F increases by $1/T$, ω increases by 2π. The DTFT expressed in terms of the $X_c(F)$ is

$$\boxed{X(\omega) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X_c\Big(\frac{\omega - 2\pi m}{2\pi T}\Big).} \tag{70}$$

The $x[n] \Longleftrightarrow X(\omega)$ relationship is that of Fourier series coefficients $x[n]$ corresponding to a periodic signal $X(\omega)$.

6.2 Wrapping a Continuous-Time Signal: $x(t) \rightarrow \tilde{x}(t)$

The frequency-domain consequences of wrapping a continuous-time signal have been explored in Section 2.4. That result is reproduced here with the appropriate change of variables,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kNT) \Longleftrightarrow \frac{1}{NT}\sum_{m=-\infty}^{\infty} X_c\Big(\frac{m}{NT}\Big)\delta\Big(F - \frac{m}{NT}\Big). \tag{71}$$

In the diagram, the frequency domain representation of the wrapped sequence is given in terms of its continuous-time Fourier series coefficients,

$$\boxed{x_m = \frac{1}{NT}X_c\Big(\frac{m}{NT}\Big).} \tag{72}$$

Note that these relationships depend only on the product NT.

6.3 Wrapping a Discrete-Time Signal: $x[n] \rightarrow \tilde{x}[n]$

The frequency-domain consequences of wrapping a discrete-time signal have been explored in Section 4.4. That result is reproduced here with the appropriate change of variables,

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n-kN] \Longleftrightarrow \frac{2\pi}{N} \sum_{m=0}^{N-1} X(\frac{2\pi m}{N}) \delta(\langle\omega\rangle_{2\pi} - \frac{2\pi m}{N}). \tag{73}$$

The discrete-time Fourier series coefficients for $\tilde{x}[n]$ are

$$X_m = \frac{1}{N} X(\frac{2\pi m}{N}). \tag{74}$$

Substituting for $X(\omega)$ from Eq. (70) gives an expression for the Fourier series coefficients directly in terms of wrapped samples of the continuous-time Fourier transform $X_c(F)$,

$$X_k = \frac{1}{NT} \sum_{m=-\infty}^{\infty} X_c(\frac{k-mN}{NT}). \tag{75}$$

In the figure, the corresponding relationship is expressed in terms of the discrete Fourier transform coefficients ($X[k] = NX_k$),

$$\boxed{X[k] = X\left(\frac{2\pi k}{N}\right).} \tag{76}$$

These coefficients are the DFT for one period of $\tilde{x}[n]$.

6.4 Sampling a Continuous-Time Periodic Signal: $\tilde{x}(t) \rightarrow \tilde{x}[n]$

The periodic signal $\tilde{x}(t)$ is represented by its Fourier series coefficients x_m in Eq. (72). The periodic discrete-time signal $\tilde{x}[n]$ is likewise represented by its Fourier series coefficients X_m in Eq. (75). The relationship between these is

$$X_k = \sum_{m=-\infty}^{\infty} x_{k-mN}. \tag{77}$$

Finally, the DFT coefficients expressed in terms of the Fourier series coefficients of $\tilde{x}(t)$ are given by

$$\boxed{X[k] = N \sum_{m=-\infty}^{\infty} x_{k-mN}.} \tag{78}$$

6.5 Frequency Domain Relationships

6.5.1 Reversibility

The diagram shows that sampling in one domain corresponds to wrapping in the other domain. The diagram shows directed arrows for the sampling and wrapping operations. Under some circumstances, one can "reverse" one of these operations.

Start from the top of the figure and follow the time domain signal on the left side. Time sampling is theoretically reversible if a time-domain signal is appropriately bandlimited, §3.1. Similarly, wrapping a time signal is reversible if the signal is time limited to less than the wrapping period. However to reach the bottom of the diagram from the top involves both sampling and wrapping. The combination is not reversible since a signal cannot be simultaneously bandlimited and time limited.

Similarly following operations on the right that go from the top of the figure to the bottom, are not exactly reversible since they involve reversing both aliasing and sampling.

6.5.2 Periodic $x(t)$

Consider a periodic continuous-time signal $x(t)$. Sampling this signal is well-defined. However, wrapping this signal can result in the sum becoming infinite. Since going from continuous-time to the DFT input $\tilde{x}[n]$ involves both sampling and wrapping, this is generally not possible for a periodic $x(t)$.

Consider

$$x(t) = e^{j2\pi F_o t}. \tag{79}$$

This periodic signal has a Fourier series with a single term and thus has a Fourier transform consisting of a single delta function at $F = F_o$. Sampling $x(t)$ at kT results in a DTFT which has a delta functions at $\omega = 2\pi F_o T + 2\pi m$. There is a single delta function in any interval of length 2π.

Now consider a more general periodic signal which has an unbounded number of harmonics,

$$x(t) = \sum_{m=-\infty}^{\infty} x_m e^{j2\pi m F_o t}. \tag{80}$$

If this signal is sampled at kT, there are two cases. If $F_o T = M/N$ where M and N are relatively prime, then the discrete-time signal will be periodic with period N. Since the Fourier series expansion for a periodic signal with period N has at most N terms, the DTFT will contain at most N delta functions in an interval of length 2π. If FoT is irrational, the DTFT can potentially contain an infinite number of delta functions in every interval of length 2π.

The conclusion is that sampled periodic signals have a DTFT consisting of a finite number of delta function per 2π interval if the sampled signal is itself periodic (requires the sampling interval be synchronized with the period), or if the periodic signal has a finite number of harmonics (Fourier series expansion with a finite number of terms).

7 Summary

These notes have shown that the Fourier transform can be applied to periodic continuous-time or discrete-time signals. This allows for a unified analysis of signals containing both non-periodic and periodic components. The second part of these notes have examined the frequency domain relationships for signals derived by sampling and/or wrapping a continuous-time signal.

Appendix A Multiplication/Convolution Relationships

Multiplication in the time domain corresponds to convolution in the frequency domain. There are four kinds of convolution depending on whether it involves continuous or discrete signals and whether the signals are periodic or not.

A.1 Convolution of Continuous Signals

The first example is the product of two continuous-time signals which corresponds to the convolution of their Fourier transforms. The second example is the convolution of two continuous-time signals which corresponds to the product of their Fourier transforms.

$$\begin{aligned} p(t) = x(t)y(t) &\Longleftrightarrow \int_{-\infty}^{\infty} X(G)Y(F-G)\,dG = P(F) \\ p(t) = \int_{-\infty}^{\infty} x(u)y(t-u)\,du &\Longleftrightarrow X(F)Y(F) = P(F) \end{aligned} \tag{81}$$

A.2 Convolution of Continuous Periodic Signals

The first example is product of two discrete-time signals which corresponds to the periodic convolution their discrete-time Fourier transforms. The second example is the periodic convolution of two continuous-time periodic signals, each with period T, which corresponds to the product of their Fourier series coefficients,

$$\begin{aligned} p[n] = x[n]y[n] &\Longleftrightarrow \frac{1}{2\pi}\int_{0}^{2\pi} X(\nu)Y(\omega-\nu)\,d\nu = P(\omega) \\ \tilde{p}(t) = \int_{0}^{T} \tilde{x}(u)\tilde{y}(t-u)\,du &\Longleftrightarrow X_k Y_k = P_k. \end{aligned} \tag{82}$$

A.3 Convolution of Discrete Signals

The first example is the product of two continuous-time periodic signals, each with period T, which corresponds to the convolution of their Fourier series coefficients. The second example is the convolution of two discrete-time signals, which corresponds to the product of their discrete-time Fourier transforms.

$$\begin{aligned} \tilde{p}(t) = \tilde{x}(t)\tilde{y}(t) &\Longleftrightarrow \sum_{l=-\infty}^{\infty} X_l Y_{k-l} = P_k \\ p[n] = \sum_{m=0}^{N-1} x[m]y[n-m] &\Longleftrightarrow X(\omega)Y(\omega) = P(\omega) \end{aligned} \tag{83}$$

A.4 Convolution of Discrete Periodic Signals

The first example is the product of two discrete-time signals, each of length N, which corresponds to the discrete periodic convolution of their discrete-time Fourier transform coefficients. The second example is the periodic convolution of two discrete-time periodic sequences, each of length N, which corresponds to the product of their discrete Fourier transform coefficients.

$$
\begin{aligned}
p[n] = x[n]y[n] &\Longleftrightarrow \frac{1}{N}\sum_{l=0}^{N-1} X[l]Y[k-l] = P[k] \\
\tilde{p}[n] = \sum_{m=0}^{N-1} \tilde{x}[m]\tilde{y}[n-m] &\Longleftrightarrow X[k]Y[k] = P[k]
\end{aligned}
\tag{84}
$$

Appendix B Continuous-Time Results Expressed in Radian Measure

In this appendix, some of the results in the main text for continuous-time signals are restated using radian frequency. Using radian frequency (ω), the Fourier transform is

$$V(\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t}\, dt. \tag{85}$$

The inverse transform is

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\omega) e^{j\omega t}\, d\omega. \tag{86}$$

The integral representation for a delta function in Eq. (11) has a 2π factor in the exponent. Absorbing this factor into the variable u, a modified integral representation is

$$\boxed{\int_{-\infty}^{\infty} e^{\pm jux}\, du = 2\pi\delta(x).} \tag{87}$$

Using this result, the Fourier transform of a periodic sequence expressed in terms of ω is (cf. Eq. (15))

$$\boxed{V_p(\omega) = 2\pi \sum_{m=-\infty}^{\infty} v_m \delta(\omega - \frac{2\pi m}{T}),} \tag{88}$$

where v_m is given in Eq. (13). The Fourier transform of the periodic impulse train is (cf. Eq. (20))

$$\boxed{\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} e^{j2\pi mt/T} \Longleftrightarrow \sum_{k=-\infty}^{\infty} e^{-jk\omega T} = \frac{2\pi}{T} \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi m}{T}).} \tag{89}$$

The Poisson sum formula for the Fourier transform with radian argument is (cf. Eq. (24))

$$\boxed{\sum_{k=-\infty}^{\infty} v(t - kT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} V(\frac{2\pi m}{T}) e^{j2\pi tm/T}.} \tag{90}$$

References

[1] D. G. Manolakis and V. K. Ingle, *Applied Signal Processing: Theory and Practice*, Cambridge University Press, 2011 (ISBN: 978-0-521-11002-0).

[2] A. Papoulis, *The Fourier Integral and its Applications*, McGraw-Hill, 1962 (ISBN: 978-0-07-048447-4).