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Real and Complex Finite Length Sequences with Linear Phase Responses

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1 Introduction

This report provides an analysis of finite-length discrete-time responses which correspond to linear-phase frequency responses. For these responses it is shown that the (complex) discrete-time response has even or odd conjugate symmetry and the DTFT (discrete-time Fourier transform) can be decomposed into a purely real response in tandem with phase terms. For real discrete-time coefficients, there are four cases: even/odd symmetry together with an even/odd number of coefficients. Then the real frequency response can be decomposed into the product of a real response which only depends on which of the four cases applies, and a real frequency response corresponding to coefficients with even symmetry and an odd number of coefficients.

In this report a discrete-time response will be shown as $g[n]$; the z-transform of $g[n]$ is $\mathcal{G}(z)$; and the frequency response (Discrete-Time Fourier Transform) is $G(\omega)$, where $G(\omega) = \mathcal{G}(e^{j\omega})$.

The frequency response of a generalized linear-phase frequency response[1] can be written as

$$G(\omega) = e^{-j\alpha\omega}\, e^{j\beta}\, B(\omega), \tag{1}$$

where $B(\omega)$ is a real function of frequency. There is a phase contribution from each of the three terms in this equation: the first term gives a linear-phase (corresponding to a delay of α samples); the second term gives a constant phase β; and the third term is a real frequency response which contributes a frequency-dependent phase of either 0 or π. The symmetry conditions requirements on the coefficients of the time response $g[n]$ such that the frequency response is of this form, will be derived. If $g[n]$ is real, it is shown that the frequency response $B(\omega)$ can be further factored if the number of coefficients is even and/or the time response is odd-symmetric.

2 Symmetry Conditions

2.1 Simplified Case

First consider a simplified case – more complexity will be added step-by-step. The time response $g[n]$ will be causal with a non-zero first coefficient $g[0]$ and non-zero last coefficient $g[N-1]$. The frequency response will be linear-phase,

$$G(\omega) = e^{-j\alpha\omega}\, B(\omega), \tag{2}$$

[1] A generalized linear-phase response is the sum of a phase which is linear with frequency and a constant phase.

where $B(\omega)$ is purely real. The inverse relationship corresponding to Eq. (2) is

$$B(\omega) = e^{j\alpha\omega} G(\omega). \tag{3}$$

The goal will be to find the requirements on the coefficients of $G(\omega)$ such that $B(\omega)$ is real.

The z-transform corresponding to $g[n]$ is

$$\mathcal{G}(z) = \sum_{n=0}^{N-1} g[n] z^{-n}. \tag{4}$$

To accommodate both even and odd numbers of coefficients, consider the up-sampling the time samples to form $g_2[n]$, [2]

$$g_2[n] = \begin{cases} g[n/2], & \text{for } n = 0, 2, 4, \ldots, 2(N-1), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

The upsampling operation inserts zero-valued samples between the coefficients, giving an odd-length sequence of length $2N-1$. The z-transform of $g_2[n]$ is $\mathcal{G}_2(z)$, where

$$\mathcal{G}_2(z) = \mathcal{G}(z^2). \tag{6}$$

Form an upsampled shifted response with the z-transform,

$$\mathcal{B}(z^2) = z^{2\alpha} \mathcal{G}(z^2), \tag{7}$$

with the corresponding pulse response $b_2[n]$.

2.2 Integer / Half-Integer Delay

Consider the case that 2α is an integer. Then the shifted upsampled sequence, denoted as $b_2[n]$, is

$$b_2[n - 2\alpha] = g_2[n], \qquad \text{for } 0 \le n \le 2(N-1). \tag{8}$$

For a frequency response $X(\omega)$ to be real-valued, the discrete-time coefficients must satisfy $x[n] = x^*[-n]$. Identifying $\mathcal{B}(z^2)$ with $\mathcal{X}(z)$, then for the frequency response $B(2\omega)$ to be real, the coeffi-

[2] This artifice is being introduced to solve that problem that for an even number of coefficients, the time response $g[n]$ cannot be shifted so as to centre it around zero.

cients must have even conjugate symmetry,

$$b_2[n] = b_2^*[-n], \qquad \text{for } -(N-1) \le n \le N-1. \tag{9}$$

Given that $b_2[n]$ is just a shifted version of $g_2[n]$, and that $g_2[n]$ is of finite length, the shift must centre $b_2[n]$ around zero. The delay α is then an integer or half-integer of the form

$$\alpha = \frac{N-1}{2}. \tag{10}$$

The upsampled coefficients $b_2[n]$ will have $N-1$ samples to the left of $n=0$ and $N-1$ samples to the right of $n=0$.

The response $g[n]$ can be expressed in terms of the non-zero coefficients of $b_2[n]$,

$$g[n] = b_2[2n-(N-1)], \qquad \text{for } n = 0, \ldots, N-1. \tag{11}$$

Using the conjugate symmetry of $b_2[n]$, $g[n]$ is conjugate-symmetric about the middle of the sequence,

$$g[n] = g^*[N-1-n], \qquad \text{for } n = 0, \ldots, N-1. \tag{12}$$

In terms of z-transforms,

$$\mathcal{G}(z) = z^{-(N-1)}\mathcal{G}^*(1/z^*). \tag{13}$$

The double conjugation (once on z and again on $\mathcal{G}(\cdot)$), leaves z not conjugated, and the coefficients $g[n]$ conjugated. The frequency response in Eq. (2) becomes

$$G(\omega) = e^{-j\omega(N-1)/2}B(\omega), \tag{14}$$

where $B(\omega)$ is real. If the coefficients $g[n]$ are real, then $B(\omega)$ is real and symmetric. If the coefficients $g[n]$ are complex, $B(\omega)$ is real but not symmetric.

2.2.1 General delay

For a general α, interpolate $g[n]$ to form a bandlimited continuous-time signal,

$$g(t) = \sum_{n=0}^{N-1} g[n]\,\text{sinc}(t-n), \tag{15}$$

where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. The discrete-time signal $g[n]$ is linear phase if and only if $g(t)$ is conjugate-symmetrical about $t = \alpha$ [1],

$$g(t+\alpha) = g^*(\alpha - t). \tag{16}$$

These terms can be expanded as follows,

$$\begin{aligned} g(t+\alpha) &= \sum_{n=0}^{N-1} g[n]\,\mathrm{sinc}(t+\alpha-n), \\ g^*(\alpha - t) &= \sum_{n=0}^{N-1} g^*[N-1-n]\,\mathrm{sinc}(t-\alpha+N-1-n). \end{aligned} \tag{17}$$

The symmetry of the $\mathrm{sinc}(\cdot)$ function has been used in the second line of this equation.

Set $g[n] = g^*[N-1-n]$, then equating $g(t+\alpha)$ and $g^*(\alpha-t)$ requires $\alpha = (N-1)/2$. Conversely, set $\alpha = (N-1)/2$, then $g[n] = g^*[N-1-n]$. No other value of α together with a finite number of coefficients will allow $g(t)$ to satisfy the conjugate symmetry requirements.

As a side note, there are bandlimited, continuous-time signals which when sampled give a *causal* infinite-extent (IIR) linear-phase discrete-time signal [1]. However, the resulting IIR response does *not* correspond to a rational z-transform.

2.3 Constant Phase Term

Fix $\alpha = (N-1)/2$, and for more generality, add a constant phase term $e^{j\beta}$ to Eq. (14),

$$G(\omega) = e^{-j\omega(N-1)/2} e^{j\beta} B(\omega). \tag{18}$$

The constant phase term is a complex constant, independent of frequency. Then

$$\mathcal{G}(z^2) = e^{j\beta} z^{-(N-1)} \mathcal{B}(z^2). \tag{19}$$

The relationship in terms of the coefficients of the time response is

$$\begin{aligned} e^{-j\beta} g[n] &= b_2[2n-(N-1)], \qquad &&\text{for } n = 0,\ldots,N-1, \\ e^{j\beta} g^*[N-1-n] &= b_2^*[N-1-2n], \qquad &&\text{for } n = 0,\ldots,N-1. \end{aligned} \tag{20}$$

Then using the conjugate symmetry of $b_2[n]$ in Eq. (9),

$$g[n] = e^{j2\beta} g^*[N-1-n], \qquad \text{for } n = 0,\ldots,N-1. \tag{21}$$

or in z-transform notation,

$$\mathcal{G}(z) = e^{j2\beta} z^{-(N-1)} \mathcal{G}^*(1/z^*). \tag{22}$$

2.3.1 Symmetry imposed by the constant phase term

The coefficient symmetries can be expressed using $\tilde{g}[n]$ and $\widetilde{\mathcal{G}}(z)$, where

$$\tilde{g}[n] = e^{-j\beta} g[n], \qquad \widetilde{\mathcal{G}}(z) = e^{-j\beta} \mathcal{G}(z). \tag{23}$$

Then

$$\tilde{g}[n] = \tilde{g}^*[N-1-n], \qquad \widetilde{\mathcal{G}}(z) = z^{-(N-1)} \widetilde{\mathcal{G}}^*(1/z^*). \tag{24}$$

If N is odd, the middle coefficient of $\tilde{g}[n]$ is real. The z-transform with coefficients $\tilde{g}[n]$ has the same zeros as the z-transform with coefficients $g[n]$.

If the phase β is equal to one of the cardinal angles $0, \pm\pi/2$, or $\pm\pi$, from Eq. (21) the symmetry for $g[n]$ is

$$g[n] = \pm g^*[N-1-n], \qquad \text{for } n = 0, \ldots, N-1. \tag{25}$$

Use the plus sign for β equal to 0 or $\pm\pi$ and the minus sign for β equal to $\pm\pi/2$.

If the coefficients are real, from Eq. (21) the term $e^{j2\beta}$ must be real, i.e., β *must* be one of the cardinal angles, see also [2, § 6.5.3].

The symmetry constraints imposed by Eq. (21) are shown schematically in Table 1.

Table 1 Coefficient symmetries for linear-phase responses. The first 3 rows show the configurations for $g[n]$ complex. The last 2 rows show the configurations for $g[n]$ real. In the table u and v are complex values; a, b, c, and r are real values.

β	N odd	N even
general	$e^{j\beta}\begin{bmatrix} u & v & r & v^* & u^* \end{bmatrix}$	$e^{j\beta}\begin{bmatrix} u & v & v^* & u^* \end{bmatrix}$
0 or $\pm\pi$	$\begin{bmatrix} u & v & r & v^* & u^* \end{bmatrix}$	$\begin{bmatrix} u & v & v^* & u^* \end{bmatrix}$
$\pm\pi/2$	$\begin{bmatrix} u & v & jr & -v^* & -u^* \end{bmatrix}$	$\begin{bmatrix} u & v & -v^* & -u^* \end{bmatrix}$
0 or $\pm\pi$	$\begin{bmatrix} a & b & c & b & a \end{bmatrix}$	$\begin{bmatrix} a & b & b & a \end{bmatrix}$
$\pm\pi/2$	$\begin{bmatrix} a & b & 0 & -b & -a \end{bmatrix}$	$\begin{bmatrix} a & b & -b & -a \end{bmatrix}$

The rows of that table can be summarized as follows.

1. For the first line in the table, the entries give the form of $g[n] = e^{j\beta}\tilde{g}[n]$. For *complex* coefficients and a general β,

$$\begin{aligned} g[n] &= e^{j2\beta} g^*[N-1-n], \qquad \text{for } n = 0, \ldots, N-1 \\ G(\omega) &= e^{j\beta} e^{-j\omega(N-1)/2} B(\omega) \end{aligned} \tag{26}$$

If the number of coefficients is odd, the middle coefficient of $g[n]$ must be of the form $re^{j\beta}$, where r is the real middle coefficient of $\tilde{g}[n]$.

For a set of complex coefficients, the response can be written as the product of the fixed phase factor and conjugate-symmetric coefficients $\tilde{g}[n]$. The value of β can be determined from $g[n]$ by calculating the complex cross-correlation coefficient between the vector of coefficients $g[n]$ and $g^*[N-1-n]$. For column vectors $\mathbf{x}$ and $\mathbf{y}$, the cross-correlation coefficient [3, § 4.1] is the complex scalar value

$$\rho(\mathbf{x},\mathbf{y}) = \frac{\mathbf{y}^H\mathbf{x}}{\sqrt{\mathbf{x}^H\mathbf{x}}\,\sqrt{\mathbf{y}^H\mathbf{y}}}, \tag{27}$$

where the superscript H denotes the conjugate (Hermitian) transpose. Associate $\mathbf{x}$ with $g[n]$ and $\mathbf{y}$ with $g[N-1-n]$. For $g[n]$ corresponding to a linear phase spectrum (i.e., satisfying Eq. (21)). this gives the cross-correlation

$$e^{j2\beta} = \cos(2\beta) + j\sin(2\beta). \tag{28}$$

The angle β can have two values, separated by π. The fixed phase term $\exp(j\beta)$ can be found by computing

$$\begin{aligned} e^{j\beta} &= \cos(\beta) + j\sin(\beta) \\ &= \pm\sqrt{\frac{1+\cos(2\beta)}{2}} + j\sqrt{\frac{1-\cos(2\beta)}{2}}. \end{aligned} \tag{29}$$

Choose the plus sign for $\sin(2\beta) \geq 0$ and the minus sign otherwise. This choice of signs gives $0 \leq \beta < \pi/2$.

2. For *complex* coefficients and $\beta = 0$ or $\beta = \pm\pi$, the coefficients have conjugate symmetry,

$$\begin{aligned} g[n] &= g^*[N-1-n], \qquad \text{for } n = 0,\ldots,N-1 \\ G(\omega) &= \pm e^{-j\omega(N-1)/2}B(\omega) \end{aligned} \tag{30}$$

If the number of complex coefficients is odd, the middle coefficient of $g[n]$ must be its own conjugate, i.e. the middle coefficient must be real.

3. For *complex* coefficients and $\beta = \pm\pi/2$, the coefficients have odd conjugate symmetry,

$$\begin{aligned} g[n] &= -g^*[N-1-n], \qquad \text{for } n = 0,\ldots,N-1 \\ G(\omega) &= \pm je^{-j\omega(N-1)/2}B(\omega) \end{aligned} \tag{31}$$

If the number of coefficients is odd, the middle coefficient of $g[n]$ must be purely imaginary or zero.

4. For *real* coefficients and $\beta = 0$ or $\beta = \pm\pi$, the coefficients have even symmetry,

$$\begin{aligned} g[n] &= g[N-1-n], \qquad \text{for } n = 0, \ldots, N-1 \\ G(\omega) &= \pm e^{-j\omega(N-1)/2} B(\omega) \end{aligned} \tag{32}$$

5. For *real* coefficients and $\beta = \pm\pi/2$, the coefficients have odd symmetry,

$$\begin{aligned} g[n] &= -g[N-1-n], \qquad \text{for } n = 0, \ldots, N-1 \\ G(\omega) &= \pm j e^{-j\omega(N-1)/2} B(\omega) \end{aligned} \tag{33}$$

   If N is odd, the middle coefficient is zero.

Note that a minus sign in the expressions for $G(\omega)$ can be absorbed into the real response $B(\omega)$. Then for real coefficients, $G(\omega)$ is a linear phase shift of a purely real response $B(\omega)$ or a linear phase shift of a purely imaginary response $jB(\omega)$.

For a real time response, the corresponding frequency response must have a phase function which is an odd function of frequency. Yet the formulations for a real odd-symmetric $g[n]$ show a constant phase shift of $\pm\pi/2$. This contradiction is resolved later when it is shown that for an odd-symmetric $g[n]$, $B(\omega)$ is zero for $\omega = 0$. An odd number of zeros at $\omega = 0$ results in a phase jump of π so that the overall phase function is an odd function of frequency.

2.4 Shifted Coefficients

The final step of generality will be a shift of K samples applied to $g[n]$,

$$h[n] = g[n-K], \qquad K \le n \le K+N-1. \tag{34}$$

The first non-zero coefficient is now at time $n = K$, and the last non-zero coefficient is at time $n = K+N-1$.

The K sample shift expressed as a z-transform is,

$$\mathcal{H}(z) = z^{-K} \mathcal{G}(z). \tag{35}$$

The z-transform $\mathcal{H}(z)$ will have the same singularities in the finite z-plane ($0 < |z| < \infty$) as $\mathcal{G}(z)$, but the shift by K samples adds K poles at $z = \infty$ ($K > 0$) or K poles at $z = 0$ ($K < 0$).

The symmetry conditions for the shifted time response are

$$h[n] = e^{j2\beta} h^*[2K+N-1-n], \qquad K \le n \le K+N-1. \tag{36}$$

In terms of the z-transform this relationship is

$$\mathcal{H}(z) = e^{j2\beta} z^{-2K} z^{-(N-1)} \mathcal{H}^*(1/z^*). \tag{37}$$

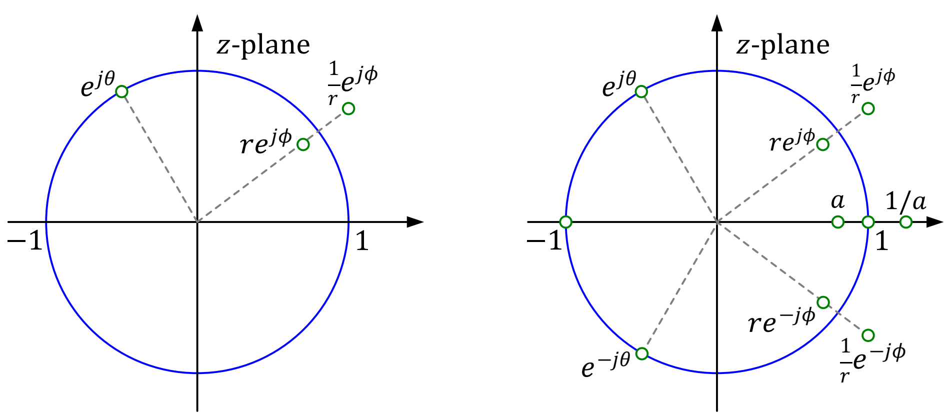
The subsequent sections revert to consideration of the coefficients $g[n]$ and its transforms.

3 Zero Symmetries of the System Response

Consider the singularities of $\mathcal{G}(z)$. This causal response has $N-1$ poles at the origin, and $N-1$ zeros in the finite z-plane .

3.1 Zero Symmetries: Complex Coefficients

From Eq. (22), if z_k is a zero of $\mathcal{G}(z)$, then its conjugate-reciprocal $1/z_k^*$ is also a zero. Write z_k as $z_k = r_k e^{j\theta_k}$. Then the conjugate-reciprocal zero is $1/z_k^* = (1/r_k)e^{j\theta_k}$. Zeros on the unit circle are their own conjugate-reciprocals, and hence can appear singly. The zero symmetries for complex coefficients are illustrated in Fig. 1(a).



(a) Complex coefficients **(b)** Real coefficients

Fig. 1 Zero symmetries

For an odd number of coefficients N, there are an even number of zeros. Since zeros off the unit circle occur in conjugate-reciprocal pairs, the number of zeros on the unit circle must be even. For an even number of coefficients, there must be an odd number of zeros on the the unit circle, i.e., there must be at least one zero on the unit circle.

The z-transform $\mathcal{G}(z)$ can be factored into first order and second order sections. A first order factor for a zero at $z = e^{j\theta}$ can be written as

$$\mathcal{G}_1(z) = 1 - e^{j\theta} z^{-1}. \tag{38}$$

Evaluating $\mathcal{G}(z)$ at $z = e^{j\omega}$, the frequency response is

$$G_1(\nu) = 2je^{-j\nu/2} \sin(\nu/2), \tag{39}$$

where $\nu = \omega - \theta$. In this form, as θ changes, the frequency response shifts along the ω axis. The linear phase term is $e^{-j\omega/2}$ and the fixed phase component is $je^{j\theta/2}$. When the zero is at $z = 1$, the fixed phase term is $e^{j\pi/2}$.

A second order factor for a pair of zeros at angle ϕ can be written as

$$\mathcal{G}_2(z) = (1 - re^{j\phi} z^{-1})(1 - (1/r)e^{j\phi} z^{-1}). \tag{40}$$

The corresponding frequency response is

$$G_2(\nu) = 2e^{-j\nu} \Big(\cos(\nu) - \frac{r^2+1}{r} \Big), \tag{41}$$

with $\nu = \omega - \phi$. As ϕ changes, the frequency response is shifted along the frequency axis. The fixed phase term is $e^{-j\omega}$ and the fixed phase term is $e^{j\phi}$.

3.2 Zero Symmetries: Real Coefficients

For real coefficients, the zeros of $\mathcal{G}(z)$ must appear in conjugate-reciprocal pairs *and* complex conjugate pairs. If $z_k = r_k e^{j\theta_k}$ is a zero of $\mathcal{G}(z)$, then so are $z_k^* = r_k e^{-j\theta_k}$, $1/z_k^* = (1/r_k)e^{j\theta_k}$, and $1/z_k = (1/r_k)e^{-j\theta_k}$. Complex zeros off the unit circle appear in fours. Complex zeros on the unit circle appear in pairs. Real zeros off the unit circle also appear in pairs. Only zeros at $z = \pm 1$ can appear singly. The zero symmetries for real coefficients are shown in Fig. 1(b).

For an odd number of coefficients N, there are an even number of zeros. The total number of zeros at $z = \pm 1$ must be even. For an even number of coefficients, there are an odd number of zeros. The total number of zeros at $z = \pm 1$ must be odd.

Using the first and second order frequency responses found for complex coefficients, the fixed phase terms terms cancel, or are already zero, with the exception of zeros at $z = 1$, in which case each such zero contributes a fixed phase of $\pi/2$.

3.3 Constrained Zeros: Real Coefficients

For real coefficients, some of the zeros are constrained to appear at $z = \pm 1$. There are four cases to consider, designated as linear-phase responses of Types I through IV in Table 2 (see also [4, § 5.7.3]). These are the cases shown in the last two rows of Table 1.

Table 2 Response type designations for linear-phase responses with real coefficients

Type	Symmetry	N
I	$g[n] = g[N-1-n]$	Odd
II	$g[n] = g[N-1-n]$	Even
III	$g[n] = -g[N-1-n]$	Odd
IV	$g[n] = -g[N-1-n]$	Even

Specializing Eq. (22) for real coefficients, write

$$\mathcal{G}(z) = \frac{1}{2}\big(\mathcal{G}(z) \pm z^{-(N-1)}\mathcal{G}(1/z)\big), \tag{42}$$

with the $+$ sign applying to even symmetry and the $-$ sign applying to odd symmetry. This relationship will be examined for $z = \pm 1$.

3.4 Zeros at $z = 1$

For $z = 1$,

$$\mathcal{G}(1) = \frac{\mathcal{G}(1)}{2}\big(1 \pm 1\big). \tag{43}$$

The response will be zero at $z = 1$ for odd symmetry. Then $\mathcal{G}(z)$ has a zero at $z = 1$ for response Types III and IV.

3.5 Zeros at $z = -1$

For $z = -1$,

$$\mathcal{G}(-1) = \frac{\mathcal{G}(-1)}{2}\big(1 \pm (-1)^{-(N-1)}\big). \tag{44}$$

The response will be zero at $z = -1$ if N is even and has an even-symmetric time response, or if N is odd and has an odd-symmetric time response. Then $\mathcal{G}(z)$ has a zero at $z = -1$ for response Types II and III.

3.6 Fixed Root Factors: Real Coefficients

Noting the fixed roots found, separating out a factor containing the fixed roots gives

$$\mathcal{G}(z) = \mathcal{Q}(z)\mathcal{P}(z), \tag{45}$$

where $\mathcal{Q}(z)$ has roots only at $z = \pm 1$. The response $\mathcal{P}(z)$, as will be shown shortly, has an odd number of even-symmetric coefficients. The results are summarized in the Table 3.

Table 3 Fixed factors for linear-phase responses with real coefficients

Type	Description	$\mathcal{Q}(z)$	No. Coef. $\mathcal{P}(z)$
I	N odd even-symmetric	1	N
II	N even even-symmetric	$1+z^{-1}$	$N-1$
III	N odd odd-symmetric	$1-z^{-2}$	$N-2$
IV	N even odd-symmetric	$1-z^{-1}$	$N-1$

Designate the number of coefficients in $\mathcal{P}(z)$ as M, then the number of coefficients in $\mathcal{Q}(z)$ is $N-M+1$. The symmetry of $\mathcal{Q}(z)$ can be expressed as

$$\mathcal{Q}(z) = \pm z^{-(N-M)}\mathcal{Q}(1/z), \tag{46}$$

where the upper sign applies when $\mathcal{G}(z)$ is even-symmetric (Types I and II), and the lower sign applies when $\mathcal{G}(z)$ is odd-symmetric (Types III and IV). Then

$$\begin{aligned}
\mathcal{P}(z) &= \frac{\mathcal{G}(z)}{\mathcal{Q}(z)} \\
&= \frac{\pm z^{-(N-1)}\mathcal{Q}(1/z)\mathcal{P}(1/z)}{\mathcal{Q}(z)} \\
&= \frac{\pm z^{-(N-1)}[\pm z^{N-M}\mathcal{Q}(z)]\mathcal{P}(1/z)}{\mathcal{Q}(z)} \\
&= z^{-(M-1)}\mathcal{P}(1/z).
\end{aligned} \tag{47}$$

In this equation, there are only two cases: $\mathcal{G}(z)$ is even-symmetric (use the plus signs) and $\mathcal{G}(z)$ is odd-symmetric (use the minus signs). Then for all response types, $\mathcal{P}(z)$ is a Type I response (odd number of coefficients and even-symmetric).

3.7 Frequency Response: Real Coefficients

Write the frequency response corresponding to $\mathcal{G}(z)$ as

$$\begin{aligned} G(\omega) &= Q(\omega)P(\omega) \\ &= e^{-j\omega(N-M)/2} e^{j\beta} Q_0(\omega) e^{-j\omega(M-1)/2} P_0(\omega) \\ &= e^{-j\omega(N-1)/2} e^{j\beta} Q_0(\omega) P_0(\omega). \end{aligned} \tag{48}$$

In the equation above, $B(\omega)$ in Eq. (18) has been expressed as the product $Q_0(\omega)P_0(\omega)$. The terms $B(\omega)$, $Q_0(\omega)$, and $P_0(\omega)$ are so-called zero-phase responses.[3] The zero-phase response $P_0(\omega)$ can be written as

$$\begin{aligned} P_0(\omega) &= \sum_{n=-(M-1)/2}^{(M-1)/2} p[n] e^{-j\omega n} \\ &= p[0] + 2 \sum_{n=1}^{(M-1)/2} p[n] \cos(\omega n). \end{aligned} \tag{49}$$

This shows that $P_0(\omega)$ is real-valued.

For each $\mathcal{Q}(z)$ in Table 3, determine the corresponding value of β and $Q_0(\omega)$. These are shown in Table 4.

Table 4 Zero-phase fixed factors for linear-phase responses with real coefficients

Type	Description	β	$Q_0(\omega)$
I	N odd even-symmetric	0	1
II	N even even-symmetric	0	$2\cos(\omega/2)$
III	N odd odd-symmetric	$\pi/2$	$2\sin(\omega)$
IV	N even odd-symmetric	$\pi/2$	$2\sin(\omega/2)$

[3] Zero-phase frequency responses are real-valued. Zero-phase responses have a phase of 0 for those frequencies where the response is positive and a phase of $\pm\pi$ where the response is negative.

3.8 Notes on the Factors of the Frequency Response

1. The $Q_0(\omega)$ term captures at most one zero at $\omega = 0$ and/or at most one zero at $\omega = \pi$. If the zeros are of odd multiplicity, one of the zeros is assigned to $Q_0(\omega)$ and the remaining zeros to $P_0(\omega)$. If the zeros are of even multiplicity, all of the zeros are assigned to $P_0(\omega)$. As a consequence, $P_0(\omega)$ can include zeros at $\omega = 0$ and $\omega = \pi$ and those zeros will be of even multiplicity.

2. Since the frequency response of responses of Type II or Type III have a null at $\omega = \pi$ due to $Q(\omega)$, they are unsuitable for the implementation of highpass responses. Since the frequency response of responses of Type III or Type IV have a null at dc due to $Q(\omega)$, they are unsuitable for the implementation of lowpass responses.

3. The zero-phase factor $P_0(\omega)$ in Eq. (49) is periodic in ω with the requisite period of 2π. However, for even response lengths (Types II and IV), the zero-phase factor $Q_0(\omega)$ in Table 4 is periodic with period 4π. It is the linear-phase term in Eq. (48) which will ensure that the overall response has period 2π. Designate the linear-phase term as $L(\omega) = e^{-j\omega(N-1)/2}$.

$$\begin{aligned} L(\omega+2\pi) &= \begin{cases} L(\omega), & N \text{ odd}, \\ -L(\omega), & N \text{ even}, \end{cases} \\ Q_0(\omega+2\pi) &= \begin{cases} Q_0(\omega), & N \text{ odd}, \\ -Q_0(\omega), & N\text{, even}. \end{cases} \end{aligned} \tag{50}$$

The product $L(\omega)Q_0(\omega)$ is periodic, with the "proper" period 2π for both odd and even N.

The plots in Fig. 2 show the effect of $Q_0(\omega)$ on the frequency response. The top pair of plots (Type I) shows the pulse response and the corresponding zero-phase frequency response $B(\omega)$. For the second pair of plots, $Q_0(\omega)$ for Type II filters is applied to the pulse response of the Type I pulse response. The resultant frequency response has a fixed null at $\omega = \pi$. For the third pair of plots, $Q_0(\omega)$ for Type III filters is applied giving a odd-symmetric pulse response and a frequency response with a fixed null at $\omega = 0$ and at $\omega = \pi$. Finally, for Type IV, the frequency response has a fixed null at $\omega = 0$.

For each response shown in Fig. 2, the number of degrees of freedom (the number of real values needed to fully describe the pulse response) when symmetries and fixed values are taken into account, is the same. In the figure, 6 values specify each of the pulse responses.

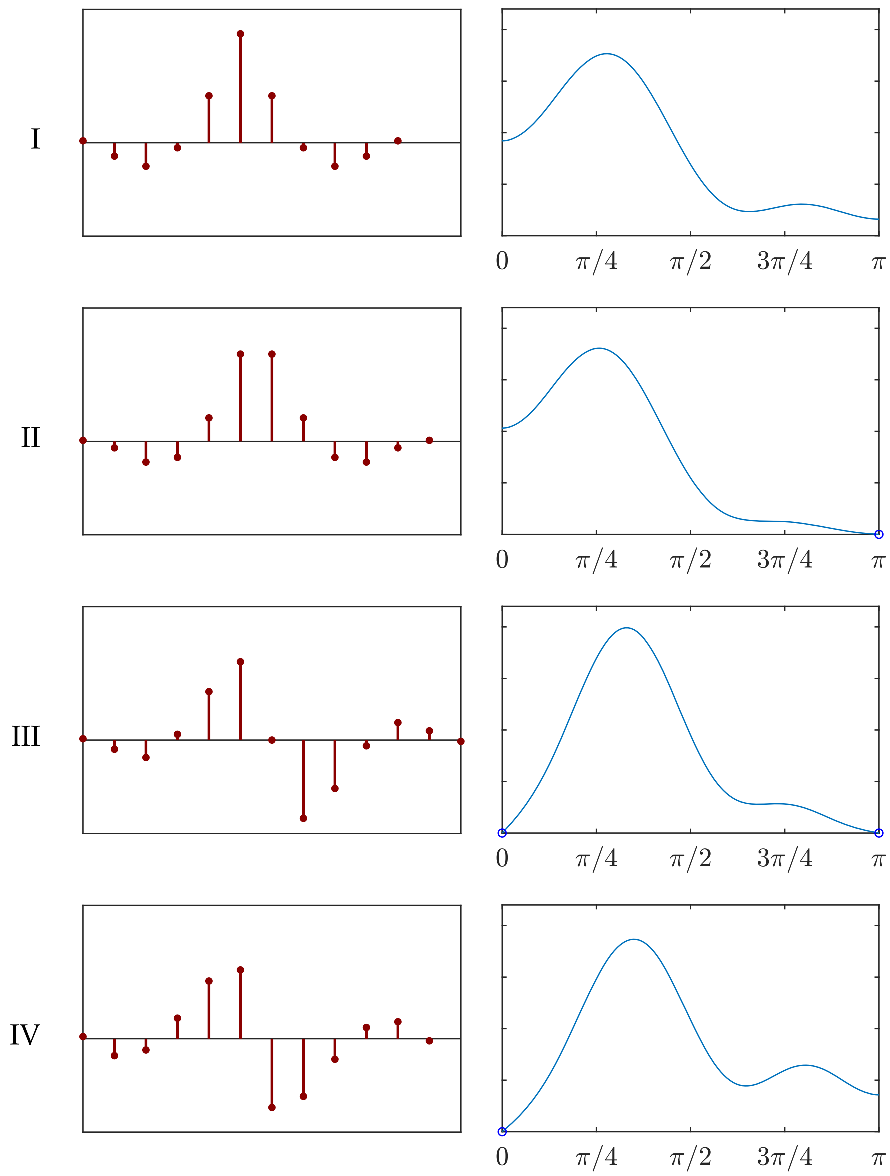


Fig. 2 Plot of the pulse response and frequency response of $B(\omega) = Q_0(\omega)P_0(\omega)$ for different types of responses. In all cases, $P_0(\omega)$ is the same, but $Q_0(\omega)$ is chosen based on the response type. The pulse responses have been normalized to unit energy. The frequencies where the frequency response is constrained to be zero are marked with circles.

Application to Filter Design

A standard filter design strategy is to specify a desired frequency response $D(\omega)$. The weighted error between the realized filter $G(\omega)$ and the desired response is

$$E_W(\omega) = W(\omega)[G(\omega) - D(\omega)], \tag{51}$$

where $W(\omega)$ is a weighting function. A function of the weighted error $E_W(\omega)$ is to be minimized during the design process. The well-known McLellan-Parks algorithm for designing linear-phase FIR filters minimizes $\max(|E_W(\omega)|$ at a dense set of frequency points [5]. The basic algorithm is formulated for Type I filters. To allow for the other types of filters, we can express $G(\omega)$ as the product $Q(\omega)P(\omega)$. The weighted error can be written as

$$\begin{aligned} E_W(\omega) &= W(\omega)[Q(\omega)P(\omega) - D(\omega)] \\ &= W(\omega)Q(\omega)[P(\omega) - D(\omega)/Q(\omega)] \\ &= \widetilde{W}(\omega)[P(\omega) - \widetilde{G}(\omega)]. \end{aligned} \tag{52}$$

By absorbing the frequency response of $Q(\omega)$ into the weighting function and the desired response, the design strategy for the Type I filter $P(\omega)$ can be used. The final filter $G(\omega)$ is then $Q(\omega)P(\omega)$.

4 Sampled Frequency Response

The goal of this section is to derive the time response $g[n]$ given samples of the zero-phase response $B(\omega)$. This is possible since the z-transform is in the form of a polynomial. The coefficients of an N term polynomial can be calculated from N distinct samples of the z-transform, for instance samples of the frequency response.

Samples of $B(\omega)$ will be the starting point. Using uniformly-spaced samples of $B(\omega)$ will simplify the calculation of $g[n]$. Let the samples of $B(\omega)$ be taken at the N points ω_k,

$$\omega_k = \frac{2\pi k}{N} + \omega_0, \qquad \text{for } k = 0, \ldots, N-1. \tag{53}$$

Now form the samples of $G(\omega_k)$ using Eq. (48). When the offset $\omega_0 = 0$, the samples $G(\omega_k)$ are the discrete Fourier transform (DFT) of $g[n]$. Then $g[n]$ can be found as the inverse DFT of $G(\omega_k)$. For the more general sample points ω_k given in Eq. (53), a modified inverse DFT formulation can be developed,

$$g[n] = \frac{1}{N} \sum_{k=0}^{N-1} G(\omega_k) e^{j\omega_k n}, \qquad \text{for } n = 0, \ldots, N-1. \tag{54}$$

For a real discrete-time response $g[n]$, $B(\omega)$ is symmetric. Then it is useful to choose ω_0 such that the samples of $B(\omega_k)$ symmetrically sample $B(\omega)$. For $\omega_0 = 0$, the samples have a DFT symmetry, $B(\omega_k) = \pm B(\omega_{N-k})$. For $\omega_0 = \pi/N$, the sampling points are offset and are symmetrical about $\omega = \pi$, giving $B(\omega_k) = \pm B(\omega_{N-1-k})$.

The modified IDFT formula Eq. (54) operates on complex-valued samples to calculate $g[n]$. Standard approaches can be used to reduce the complexity of the computations, with the resulting formulas depending on the type of filter (I, II, III, or IV) and the sampling pattern ω_k. Reference [6, § 10.2.3] applies these formulas to design linear-phase FIR filters specified by the samples $B(\omega_k)$. Examples of lowpass filters designed with this approach are given. The passband samples are set to a constant value and the stopband samples are set to zero. The transition region samples are set to the optimized values determined in [7].

5 Summary

For complex coefficients,

$$G(\omega) = e^{j\beta} e^{-j\omega(N-1)/2} B(\omega), \tag{55}$$

where $B(\omega)$ is real. The coefficients of the generalized linear-phase response obey

$$g[n] = e^{j2\beta} g^*[N-1-n], \qquad \text{for } n = 0, \ldots, N-1. \tag{56}$$

For real coefficients, the response $B(\omega)$ can be expressed as the product of two zero-phase responses,

$$G(\omega) = e^{j\beta} e^{-j\omega(N-1)/2} Q_0(\omega) P_0(\omega). \tag{57}$$

The term $Q_0(\omega)$ is a real response which depends *only* on whether the response is even or odd length, and whether the response is symmetric or anti-symmetric. The factor $P_0(\omega)$ is real. For real coefficients, β is restricted to be 0, $\pm\pi$, or $\pm\pi/2$, giving

$$g[n] = \pm g[N-1-n], \qquad \text{for } n = 0, \ldots, N-1. \tag{58}$$

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